SMCS

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S. No.	Date	Title	Page No.	Teacher's Sign./ Remarks

UNIT-4

DISCRETE PARAMETER MARKOV CHAINS:

- Markov Process:
  Markov Process is a stochastic process whose probability distribution depend only on the present state and not on how the process arrived in that state.

- Markov Chain:
  If we assume that the state space, I is discrete, then the Markov Process is known as Markov chain.

- Discrete-Parameter Markov Chain:
  If we assume that the parameter space, T is discrete, then the Markov process is known as Discrete-Parameter Markov Chain.

→ Let us define Random variables:
$X_0, X_1, X_2, \ldots, X_n, \ldots$
at time steps $0, 1, 2, \ldots$ respectively.
i.e $X_0 = 0$
$X_1 = 1$
$\vdots$
$X_n = n$

→ If $X_n = j$, then the state of the system at time step n is j.

→ According to Markov Property,
$$P(\underbrace{X_n = i_n}_{\text{future value}} \mid \underbrace{X_0 = i_0, X_1 = i_1, \ldots}_{\text{past values}}, \underbrace{X_{n-1} = i_{n-1}}_{\text{present value}}) = P(\underbrace{X_n = i_n}_{\text{future value}} \mid \underbrace{X_{n-1} = i_{n-1}}_{\text{present value}})$$

This statement implies that given the "present" state of the system, the "future" is independent of its "past".

→ The pmf $p_j(n)$ of the random variable X_n is:
$$\boxed{p_j(n) = P(X_n = j)}$$

i.e probability in state j at time step n is $p_j(n)$

→ The conditional pmf $p_{jk}(m,n)$ of random variable X_n is:
$$\boxed{p_{jk}(m,n) = P(X_n = k \mid X_m = j)}$$

This ~~is known as~~ denotes the probability that the process makes a transition from state j at step m to state k at step n.
Thus $p_{jk}(m,n)$ is known as the transition probability function of Markov chain.

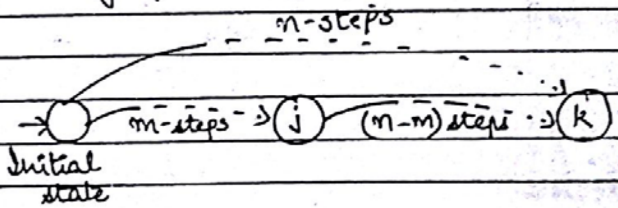
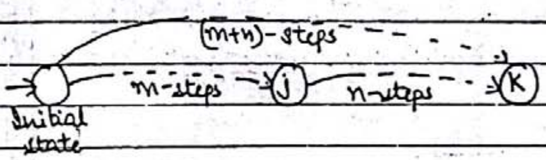


Fig. Representation of $p_{jk}(m,n)$.

→ **Homogeneous Markov Chain:**

A Markov chain for which $p_{jk}(m,n)$ depends only on the difference $(n-m)$ is called as Homogeneous Markov chain.

$$p_{jk}(n) = P(X_{m+n}=k \mid X_m = j)$$



It denotes the transition from state j to state K in n steps.

The pmf $p_{jk}(n)$ denotes the n-step transition probabilities.

Special cases:

1. 0-step transition probability:

$$p_{jk}(0) = \begin{cases} 1, & \text{if } j=k \\ 0, & \text{otherwise} \end{cases}$$

2. 1-step transition probability:

$$p_{jk} = P(X_n = k \mid X_{n-1} = j), \quad n \geq 1.$$

→ **Transition Probability Matrix:**

The one-step transition probabilities are compactly specified in the form of a Transition Probability Matrix:

$$P = [p_{ij}] = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

The entries of the Matrix P satisfy the following 2 properties:

i) $0 \leq p_{ij} \leq 1$

ii) $\sum_j p_{ij} = 1$

Special case: Stochastic Matrix

Any transition probability square matrix having non-negative entries with row sums all equal to unity is called STOCHASTIC MATRIX.

COMPUTATION OF n-STEP TRANSITION PROBABILITIES:

- We are interested in evaluating the n-step transition probability $p_{ij}(n)$ from the one-step transition probabilities $p_{ij}(1) = p_{ij}$.

- By Homogenous Markov Chain, we know:
  $p_{ij}(n) = P(X_{m+n} = j \mid X_m = i)$

- Now, we can evaluate the n-step transition probability $p_{ij}(n)$ from one-step transition probability $p_{ij}(1)$ using CHAPMAN-KOLMOGOROV EQUATION:

$$p_{ij}(m+n) = \sum_{k} p_{ik}(m) \cdot p_{kj}(n)$$

It denotes the probability that the process goes to some state k at m^{th} step, given that $X_0 = i$, is $p_{ik}(m)$;
and the probability that the process goes to state j at step $(m+n)$, given that $X_m = k$, is $p_{kj}(n)$.

Representation:

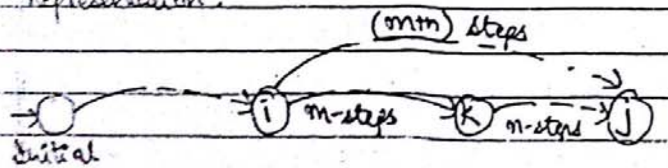


fig Chapman Kolmogorov Equation

example: $p_{02}(3+7) = p_{00}(3)\,p_{02}(7) + p_{01}(3)\,p_{12}(7) + p_{02}(3)\,p_{22}(7)$

Theorem 7.1 Given a two state Markov Chain with the transition probability matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}, \quad 0 \le a, b \le 1, \quad |1-a-b| < 1$$

the n-step transition probability matrix $P(n) = P^n$ is given by:

$$P(n) = \begin{bmatrix} \dfrac{b+a(1-a-b)^n}{a+b} & \dfrac{a-a(1-a-b)^n}{a+b} \\[2ex] \dfrac{b-b(1-a-b)^n}{a+b} & \dfrac{a+b(1-a-b)^n}{a+b} \end{bmatrix}$$

Proof:

$$P(1) = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-a & a \\ 1 & b & 1-b \end{array}$$

$p_{00}(1) = 1-a$ $\qquad p_{01}(1) = a$

$p_{10}(1) = b$ $\qquad p_{11}(1) = 1-b$

But,

$p_{00}(n) = ?$ $\qquad p_{10}(n) = ?$

$p_{01}(n) = ?$ $\qquad p_{11}(n) = ?$

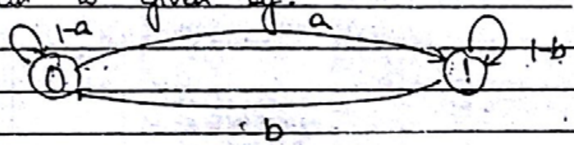
$$P(n) = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & p_{00}(n) & p_{01}(n) \\ 1 & p_{10}(n) & p_{11}(n) \end{array}$$

$P(n) = P(n-1) \cdot P(1)$

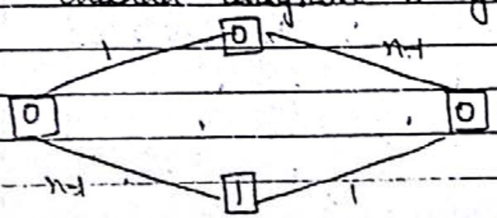
$$\begin{bmatrix} P_{00}(n) & P_{01} \\ P_{10}(n) & P_{11} \end{bmatrix} = \begin{bmatrix} P_{00}(n-1) & - \\ P_{10}(n-1) & - \end{bmatrix} \begin{bmatrix} P_{00}(1) & - \\ - & - \end{bmatrix}$$

$P_{00}(n) = P_{00}(n-1) \cdot P_{00}(1) + P_{10}(n-1) \cdot P_{10}(1)$

The state diagram of 2-state Markov chain is given by:



The chained diagram is given as:



$P_{00}(n) = P_{00}(1) \cdot P_{00}(n-1) + P_{01}(n-1) \cdot P_{10}(1)$

$\boxed{P_{00}(n) = (1-a) \cdot P_{00}(n-1) + b P_{01}(n-1)}$ — (I)

We know,

$P_{00}(n-1) + P_{01}(n-1) = 1$

Thus, $\boxed{P_{01}(n-1) = 1 - P_{00}(n-1)}$ — (II)

put (II) in (I)

$P_{00}(n) = (1-a) P_{00}(n-1) + b[1 - P_{00}(n-1)]$

$= (1-a) P_{00}(n-1) + b - b P_{00}(n-1)$

$\boxed{P_{00}(n) = (1-a-b) P_{00}(n-1) + b}$ — (III)

using (III), we get

$\boxed{P_{00}(n-1) = (1-a-b) P_{00}(n-2) + b}$ — (IV)

put (IV) in (III)

$P_{00}(n) = (1-a-b)[(1-a-b) P_{00}(n-2) + b] + b$

$\boxed{P_{00}(n) = (1-a-b)^2 P_{00}(n-2) + b(1-a-b) + b}$ — (V)

using (III)

$\boxed{P_{00}(n-2) = (1-a-b) \cdot P_{00}(n-3) + b}$ — (VI)

put (VI) in (V)

$P_{00}(n) = (1-a-b)^2 [(1-a-b) P_{00}(n-3) + b] + b(1-a-b) + b$

$\boxed{P_{00}(n) = (1-a-b)^3 P_{00}(n-3) + b(1-a-b)^2 + b(1-a-b) + b}$

In General,

$$P_{00}(n) = (1-a-b)^{n-1} P_{00}(n-n+1) + b(1-a-b)^{n-2} + \cdots + b(1-a-b)^2 + b(1-a-b) + b.$$

$$P_{00}(n) = (1-a)(1-a-b)^{n-1} + b\sum_{i=0}^{n-2}(1-a-b)^i$$

$$= (1-a)(1-a-b)^{n-1} + b\left[\frac{1-(1-a-b)^{n-1}}{\cancel{1}-\cancel{1}+a+b}\right]$$ (by G.P.)

$= \text{I}^{st}$ term

$P_{01} = 1 - \text{I}^{st}$ term

$= 1 - P_{00}(n) = \text{II}^{nd}$ term.

$$= \frac{a - a(1-a-b)^n}{a+b}$$

Now,

$\det(P^n) = P_{00}(n) - P_{10}(n)$

$\det(P^n) = (1-a-b)^n$

$P_{00}(n) - P_{10}(n) = (1-a-b)^n$

$P_{10}(n) = P_{00}(n) - (1-a-b)^n$

$= \text{III}^{rd}$ term

$P_{11}(n) = 1 - \text{III}^{rd}$ term

$= \text{IV}^{th}$ term

STATE CLASSIFICATION & LIMITING DISTRIBUTIONS:

STATE: The state of a system represents its behavior at a particular instant of time.

→ A state j is said to be accessible from state i, if there exists an integer $n \geq 1$, such that $P_{ij}^{(n)} > 0$.

→ Let $f_{ij}^{(m)}$ be the probability that a Markov chain starting with state i will reach state j for the first time at m^{th} step.

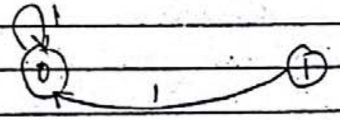
→ Now, if $P_{ij}^{(n)} > 0$, then $f_{ij}^{(n)}$ is also +ve.

→ If the state i is accessible from state j and state j is accessible from state i then i and j are said to be communicate with each other denoted as $i \leftrightarrow j$.

State Classification:

i) Transient or Non-Recurrent state:

A state ~~a~~ i is said to be transient or non-recurrent iff there is a positive probability that the process will not return to this state.

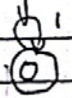


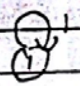
Let X_{ji} be the no. of visits to the state i starting at j. Then,

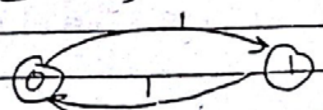
$$E[X_{ji}] = \sum_{n=0}^{\infty} P_{ji}(n)$$

ii) Recurrent State:

A state i is said to be recurrent iff starting from state i, the process eventually returns to state ~~j~~ i with probability one.

a) 

b) 

c) 

and,

$$E[X_{ji}] = \sum_{n=0}^{\infty} P_{ji}(n)$$

→ Let $f_{ij}(n)$ be the conditional probability that the first visit to state j from state i occurs in exactly n steps.

→ If i=j, then $f_{ii}(n)$ denotes the probability that the first return to state i occurs in exactly n steps.

→ ∴ these 2 probabilities are related as:

$$\boxed{P_{ij}(n) = \sum_{k=1}^{n} f_{ij}(k)\, P_{jj}(n-k)} \quad , \quad n \geq 1$$

→ Let f_{ij} denote the probability of ever visiting state j, starting from state i. Then:

$$\boxed{f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)}$$

→ It means that state i is recurrent if $f_{ii}=1$ and state i is transient if $f_{ii}<1$.

→ If $f_{ii}=1$, then the mean recurrence time of state i is denoted as:-

$$\boxed{\mu_i = \sum_{n=1}^{\infty} n f_{ij}(n)}$$

A recurrent state is further classified as:

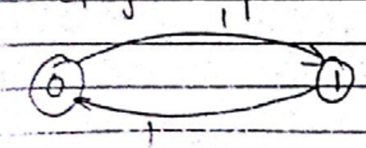
a) Recurrent non-null (Positive recurrent) State:
A recurrent state i is said to be Recurrent non-null if its mean recurrence time μ_i is finite.

b) Recurrent null state:
A recurrent state i is said to be Recurrent null if its mean recurrence time μ_i is infinite.

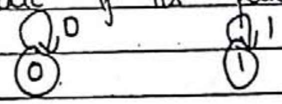
→ Period of state: If the Markov chain starting with i can return to i in n steps, where greatest common divisor is d_i, then d_i is said to be period of state i.

(3) Periodic state:
We know, for a recurrent state i, $p_{ii}(n) > 0$ for some $n \geq 1$.

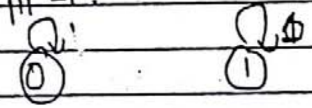
Thus, a recurrent state i is said to be periodic, if its period $d_i > 1$.



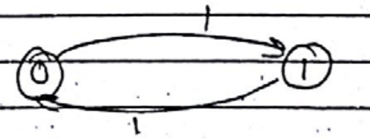
(4) Aperiodic state:
A recurrent state i is said to be aperiodic if its period $d_i = 1$.



(5) Absorbing State:
A state i is said to be absorbing state iff $p_{ii} = 1$.



(6) Irreducible Markov Chain:
- A Markov chain is said to be irreducible if every state can be reached from every other state in a finite no. of steps.
- Thus, for all $i, j \in I$, there is an integer $n \geq 1$ such that $p_{ij}(n) > 0$.



- All the states of an irreducible markov chain are of same type.
- If one state of an irreducible markov chain is aperiodic, then so are all the states and such a Markov chain is called Aperiodic Markov Chain.

Limiting Distributions:

- For a Markov chain as $n \to \infty$, i.e. n-step, the n-step transition probabilities $p_{ij}(n)$ becomes independent of both n and i. Thus, all rows of matrix P^n converge toward a common limit.

- Now, using the definition of $p_{ij}(n)$ and the theorem of total probability, we have:

$$p_j(n) = P(X_n = j) = \sum_i p_i(0)\, p_{ij}(n) \quad \text{— (1)}$$

- We know, $p_{ij}(n)$ is independent of n and i, so we conclude that $p_j(n)$ approaches a constant as $n \to \infty$.

- We denote the limiting state probabilities by:

$$v_j = \lim_{n\to\infty} p_j(n) \quad , \; j = 0, 1, \dots \quad \text{— (2)}$$

- Now,
The n-step transition probabilities $p_{ij}(n)$ of finite, irreducible, aperiodic Markov chains is independent of n & i as $n \to \infty$
Let $\quad q_j = \lim_{n\to\infty} p_{ij}(n) \quad$ — (3)

- So, eq^n (2) becomes

$$v_j = \lim_{n\to\infty} p_j(n) = \lim_{n\to\infty} \sum_i p_i(0) \cdot p_{ij}(n) \quad (\text{using eq}^n\ (1))$$

$$= \sum_i p_i(0) \left[\lim_{n\to\infty} p_{ij}(n)\right]$$

$$= \sum_i p_i(0) \cdot q_j \quad \text{— using eq}^n\ (3)$$

$$= q_j \sum_i p_i(0)$$

$$= q_j \quad \left(\because \sum_i p_i(0) = 1\right)$$

$$= \lim_{n\to\infty} p_{ij}(n) \quad \text{— (4)}$$

- This eq^n represents that P^n converges to matrix V (with identical rows $v = [v_0, v_1, \dots]$) as $n \to \infty$.

- Let us assume that for a given Markov chain, the limiting probabilities v_j exist for all states $j \in I$
i.e. $\sum_{j\in I} v_j \leq 1$

- Furthermore, either all $v_j = 0$
i.e. chain with an infinite no. of states
or $\sum_{j \in I} v_j = 1$

- If $\sum_{j\in I} v_j = 1$, the numbers $v_j, j \in I$ are said to form a steady state distribution.

- Thus we require that
i) Limiting probabilities exist
ii) Limiting distributions are independent of the initial state.
iii) Limiting distributions form a probability distribution

- The probability v_j is sometimes called as long-run proportion of time the Markov chain spends in state j.

- Now from theorem of total probability, we have

$$p_j(n) = \sum_i p_i(n-1)\, p_{ij}$$

- Then $\lim_{n\to\infty} p_j(n) = v_j = \sum \lim_{n\to\infty} p_i(n-1)$

we get,

$$\boxed{v_j = \sum_i v_i\, p_{ij}} \quad , j = 0, 1, 2$$

or in matrix notation

$$\boxed{v = vP}$$

Since v is a probability vector,

$$\therefore \boxed{v_j \geq 0 \quad , \quad \sum_j v_j = 1}$$

DISTRIBUTION OF TIMES BETWEEN STATE CHANGES:

- We know that a Markov chain has an interesting property that its future behavior depends only on the present state.

- Assume that the state at the n^{th} step is $X_n = i$, and the state at the next step is $X_{n+1} = j$. This state should depend on the current state i and not on the time that the chain has spent in the current state.

- Let us define a Random variable T_i denoting the time the Markov chain spends in state i during a single visit to state i.
The distribution of T_i should be memoryless for $\{X_n, n = 0, 1, \ldots\}$ to form a homogeneous Markov chain.

- Now, given that the chain has just entered state i at the n^{th} step and it will remain in this state at the next step with prob. P_{ii} and it will leave the state at the next step with prob. $\sum_{j\neq i} P_{ij} = 1 - P_{ii}$

- Now, if the next state is also i i.e $X_{n+1} = i$, then the same 2 choices are available at the next step also.

- Also, the prob. of events at (n+1)st step are independent of the events at n^{th} step, because $\{X_n\}$ is a Markov chain.

~~Therefore, we have a sequence of Bernoulli trials with each trial having 2 possibilities:~~

- ∴, we have a sequence of Bernoulli trials with the prob of success $1-p_{ii}$, where success is defined to be the event that the chain leaves the state i.

- Let $T_i = n$ denoting n trials upto and including 1^{st} success.
  Thus, T_i has geometric distribution, so that

$$\boxed{P(T_i = n) = (1-p_{ii})\, p_{ii}^{\,n-1}}, \quad i \in I \quad -(1)$$

- The expectation is given by:

$$\boxed{E[T_i] = \frac{1}{1-p_{ii}}}, \quad i \in I \quad -(2)$$

It denotes the expected no. of steps the chain spends in state i, per visit to state i.

- and, corresponding variance is given by:

$$\boxed{Var[T_i] = \frac{p_{ii}}{(1-p_{ii})^2}}, \quad i \in I - (3)$$

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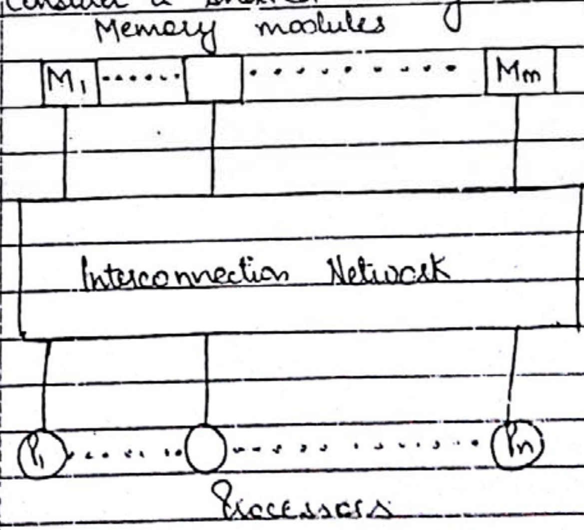
IRREDUCIBLE FINITE CHAINS WITH APERIODIC STATES:

→ Consider a shared memory multiprocessor system.

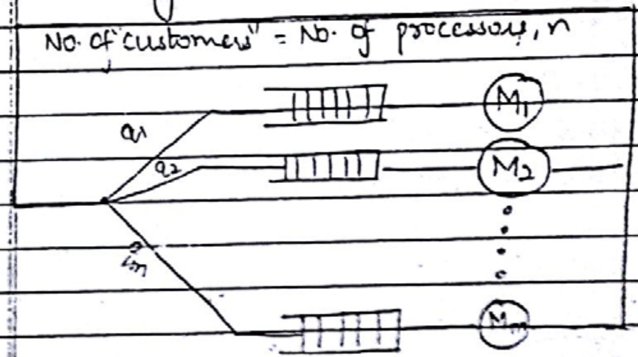


→ The fig. shows a multiprocessor S/m with shared memory in which each processor can share the entire memory space.
But the price of ~~contention~~ sharing is the contention for the shared resource.
So the memory is splitted into modules which can be accessed independently & concurrently with other modules.

→ If more than one processor tries to access the same memory module, only 1 process can be granted access, while other processors must wait for their turn in a queue.

→ We make the following assumptions:
i) Time to complete a M/m access is constant
ii) all modules are synchronized
iii) processors are fast enough to generate a new request as soon as their current request is satisfied.
iv) A processor cannot generate a new request when it is waiting for the current request to be completed.

→ Thus, the operation of the S/m can be visualized as ~~to~~ a discrete parameter queueing network as:



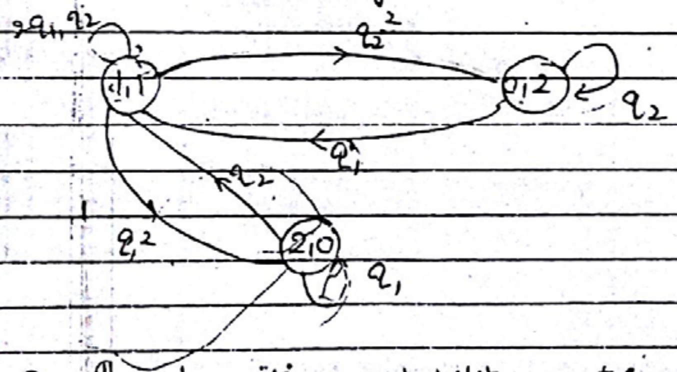
→ The memory modules are the servers and there are n processors constituting the Network.

→ Let q_i denotes the probability that a processor generated request is directed at memory module i.
Thus, $\boxed{\sum_{i=1}^{m} q_i = 1}$

→ Now,
Consider a s/m with 2 memory modules and 2 processors.
Let N_i denotes the no. of processors waiting ~~there N_1 & $N_2 \geq 0$~~ or being served at module i $(i=1,2)$
So, $N_i \geq 0$
$\therefore N_1 + N_2 = 2$

Let (N_1, N_2) denotes the state of the s/m.
So, state space $I = \{(1,1), (0,2), (2,0)\}$

→ The state diagram is shown as:



→ The transition probability matrix of this chain is given as:

	(1,1)	(0,2)	(2,0)
(1,1)	$2q_1q_2$	q_2^2	q_1^2
(0,2)	q_1	q_2	0
(2,0)	q_2	0	q_1

→ The steady state vector $v = (v_{(1,1)}, v_{(0,2)}, v_{(2,0)})$ can be calculated as:

$$v = vP \quad \text{and} \quad \sum_{(i,j) \in I} v_{(i,j)} = 1$$

$$v_{(1,1)} = 2q_1q_2 v_{(1,1)} + q_1 v_{(0,2)} + q_2 v_{(2,0)} \quad -(1)$$

$$v_{(0,2)} = q_2^2 v_{(1,1)} + q_2 v_{(0,2)} \quad -(2)$$

$$v_{(2,0)} = q_1^2 v_{(1,1)} + q_1 v_{(2,0)} \quad -(3)$$

and, $v_{(1,1)} + v_{(0,2)} + v_{(2,0)} = 1$

Thus, $v_{(2,0)} = \dfrac{q_1^2}{1-q_1} v_{(1,1)}$

$v_{(0,2)} = \dfrac{q_2^2}{1-q_2} v_{(1,1)}$

$$\Rightarrow V_{11} = \frac{1}{1 + \frac{q_1^2}{1-q_1} + \frac{q_2^2}{1-q_2}} = \frac{q_1q_2}{1-2q_1q_2}$$

→ Let B be a random variable denoting No. of m/m requests completed per m/m cycle

Let $E[B]$ denotes the average no. of m/m requests completed per m/m cycle.

Then, the conditional expectation of B is:

$E[B \mid \text{system in state } (1,1)] = 2$
$E[B \mid \text{system in state } (2,0)] = 1$
$E[B \mid \text{system in state } (0,2)] = 1$.

Now, using the total expectation is given as:

$$E[B] = 2V_{(1,1)} + V_{(0,2)} + V_{(2,0)}$$
$$\left[2 + \frac{q_1^2}{1-q_1} + \frac{q_2^2}{1-q_2}\right] V_{(1,1)}$$

$$\boxed{E[B] = \frac{1-q_1q_2}{1-2q_1q_2}}$$

DISCRETE-PARAMETER BIRTH-DEATH PROCESS:

Consider a discrete-parameter markov chain with one-step transition to nearest neighbours only, i.e. $p_{ij} = 0 \quad |i-j| > 1$

Let: $b_i = p_{i,i+1}$, $i \geq 0$ {prob. of birth in state i}
$d_i = p_{i,i-1}$, $i \geq 1$ {prob. of death in state i}
$a_i = p_{i,i}$, $i \geq 0$

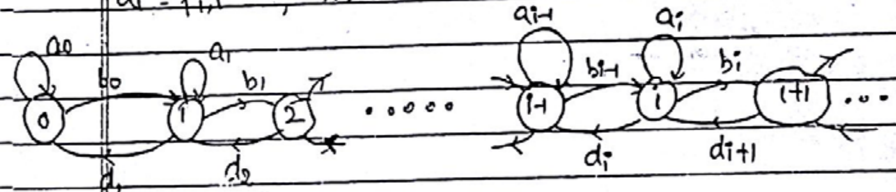


fig. State diagram of birth-death process

Now, $(a_i + b_i + d_i) = 1$ for all i. —(i)
Thus, the infinite dimensional matrix P is given by:

$$P = \begin{bmatrix} a_0 & b_0 & 0 & \cdot & - & - & - \\ d_1 & a_1 & b_1 & \cdot & - & - & - \\ 0 & d_2 & a_2 & b_2 & & & \\ \cdot & 0 & & & & & \\ & & & & a_{i-1} & b_{i-1} & \\ & & & & d_i & a_i & b_i \\ & & & & 0 & d_{i+1} & a_{i+1} \; b_{i+1} \end{bmatrix}$$

Teacher's Signature

(i) If we assume that $b_i > 0$ and $d_i = 0$ for all i, then all the states of Markov chain are <u>TRANSIENT</u>.

(ii) If we assume that $b_i = 0$ and $d_i > 0$ for all i, then all the states are transient except the state labeled 0, which is called an <u>ABSORBING STATE</u>.

(iii) If we assume $0 < b_i, d_i < 1$ for $i \geq 1$ and $b_0 > 0$, then the Markov chain is <u>IRREDUCIBLE</u> and <u>APERIODIC</u>.

Now, to compute the steady-state probability vector v, we use:-

$$v = vP$$

$$\begin{array}{c|cccccccc} & v_0 & v_1 & v_2 & \cdots & & & & \\ \hline v_0 & a_0 & b_0 & \cdots & & & & & \\ v_1 & d_1 & a_1 & b_1 & \cdots & & & & \\ v_2 & 0 & d_2 & a_2 & b_2 & & & & \\ & & 0 & & & & & & \\ & & & & & a_{i-1} & b_{i-1} & & \\ & & & & & d_i & a_i & b_i & \\ & & & & & 0 & d_{i+1} & a_{i+1} & b_{i+1} \end{array}$$

$$v_0 = a_0 v_0 + d_1 v_1 \quad \text{— (ii)}$$

$$v_1 = b_0 v_0 + a_1 v_1 + d_2 v_2$$

$\&$ so.

In general, for v_i

$$v_i = b_{i-1} v_{i-1} + a_i v_i + d_{i+1} v_{i+1} \quad i \geq 1 \text{ — (iii)}$$

also,

$$a_i = 1 - b_i - d_i \quad \text{from (i)}$$

Put above eq^n in (ii)

$$v_i = b_{i-1} v_{i-1} + (1 - b_i - d_i) v_i + d_{i+1} v_{i+1}$$

$$v_i = b_{i-1} v_{i-1} + v_i - b_i v_i - d_i v_i + d_{i+1} v_{i+1}$$

$$b_i v_i - d_{i+1} v_{i+1} = b_{i-1} v_{i-1} - d_i v_i$$

So, $b_i v_i - d_{i+1} v_{i+1}$ is independent of i

but, $b_0 v_0 - d_1 v_1 = (1 - a_0) v_0 - d_1 v_1 = 0$ (from eq^n ii)

$$\therefore \boxed{v_i = \frac{b_{i-1}}{d_i} v_{i-1} \quad \therefore \quad v_i = \prod_{j=1}^{i} \frac{b_{j-1}}{d_j} v_0}$$

now, using condition $\sum_{i \geq 0} v_i = 1$, we get

$$\boxed{v_0 = \frac{1}{\sum_{i \geq 0} \prod_{j=1}^{i} \frac{b_{j-1}}{d_j}}}$$

THE M/G/1 QUEUING SYSTEM :-

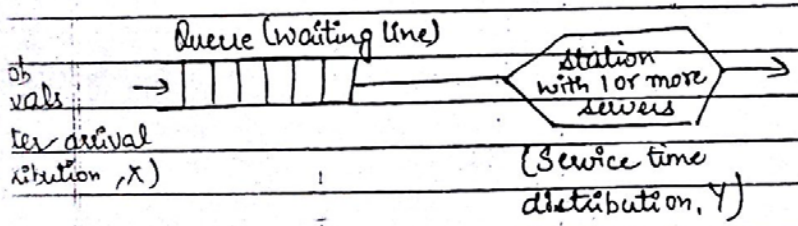
A Queue is generated when jobs arrive at a computing center to receive service.

We use the notation X/Y/Z to describe the Queuing System.

Let X denotes the inter-arrival time distribution of jobs in the system.

Let Y denotes the Service-time distribution of services in the system.

Let Z denotes the no. of Servers.



We use the following symbols :-

M : for exponential distribution

D : for deterministic or constant interarrival in service time

E_k : for K-stage Erlang distribution.

H_k : for K-stage hyperexponential distribution

G : for general distribution.

GI : for general independent inter-arrival times.

Thus, M/G/1 : denotes a single-server queue with exponential inter-arrival times and general service-time distribution.

M/M/1 : denotes a single-server queue with exponential inter-arrival times and exponential service-time distribution.

M/D/1 : denotes a single-server queue, with exponential ~~distribution~~ inter-arrival times and service times are assumed to be a constant.

{ discrete state, continuous parameter

Let $N(t)$ denotes the no. of jobs in the system at time t.
If $N(t) \geq 1$, then a job is in service.

Also, $N(t)$ requires the knowledge of time spent by the job in service in order to predict the future behavior of the system.
This means that the stochastic process $\{N(t) \mid t \geq 0\}$ is NOT a MARKOV CHAIN.

Now, let t_n $(n = 1, 2, \ldots)$ be the time of departure of n^{th} job.
Let X_n be the no. of jobs in the system at time t_n;
ie $X_n = N(t_n)$, $n = 1, 2, \ldots$ — (1)

Thus, the stochastic process $\{X_n, n = 1, 2, \ldots\}$ is now a discrete-parameter Markov chain (called as IMBEDDED MARKOV CHAIN) of the continuous parameter stochastic process $\{N(t) \mid t \geq 0\}$

It can be shown that the limiting distribution of the no. of jobs $N(t)$ ~~observed~~ observed at any arbitrary point in time is identical to the limiting distribution of the no. of jobs observed at the departure, ie

$j = i - 1 + Y_{n+1}$, $j \geq i-1$

$$\lim_{t \to \infty} P[N(t) = k] = \lim_{n \to \infty} P(X_n = k) \quad \text{— (2)}$$

Now, Let Y_n be the no. of jobs arriving during the service time of n^{th} job.

Then, $$X_{n+1} = \begin{cases} X_n - 1 + Y_{n+1} & \text{if } X_n > 0 \\ Y_{n+1} & \text{if } X_n = 0 \end{cases}$$

As, Y_{n+1} is independent of $X_1, X_2, \ldots, X_n$, it follows that given the value of X_n, we need not to know the values of $X_1, X_2, \ldots, X_{n-1}$, in order to determine the probability of X_{n+1}.
Thus, $\{X_n, n = 1, 2, \ldots\}$ is a Markov chain.

And, the transition probabilities of the Markov chain are obtained as :-

$$p_{ij} = P(X_{n+1} = j \mid X_n = i)$$

$$= \begin{cases} P(Y_{n+1} = j - i + 1), & \text{if } i \neq 0,\ j \geq i-1 \\ P(Y_{n+1} = j), & \text{if } i = 0,\ j \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The BIRTH AND DEATH PROCESSES:-

→ The Birth and Death process is a continuous parameter homogeneous Markov chain $\{X(t), t \geq 0\}$ with the state space $\{0, 1, 2, \ldots\}$ and constants $\lambda_i (i = 0, 1, \ldots)$ and $\mu_i (i = 1, 2, \ldots)$.

→ Transition rates are given by:-

i) $q_{i,i+1} = \lambda_i$

Here, $\lambda_i (i \geq 0)$ is the birth rate at which birth occurs in state i

ii) $q_{i,i-1} = \mu_i$

Here, $\mu_i (i \geq 0)$ is the death rate at which death occurs in state i.

~~Where~~

iii) $q_i = \lambda_i + \mu_i$

iv) $q_{ij} = 0$ for $|i-j| > 1$

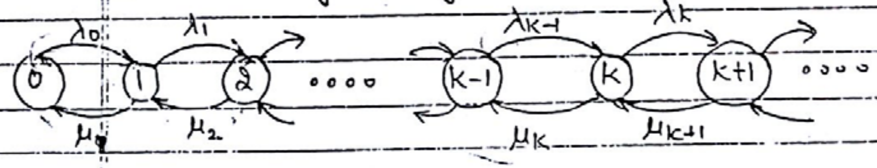
→ These birth and death rates are independent of time

→ These rates are assumed to depend only on state i.

→ In a given state, births and deaths occur independently of each other.

→ Only "one-step" transitions are allowed

State Diagram of birth-death processes:



Now,

The process will be in state K at time t+h, if one of the following mutually exclusive and collectively exhaustive events occurs:-

1. The system is in state K at time t, and no changes of state occur in the interval (t, t+h]; the associated conditional prob. is :-

$p_{K,K}(t, t+h) = 1 - q_K(t) \cdot h + o(h)$
$= 1 - (\lambda_K + \mu_K) \cdot h + o(h)$

2. The system is in state K-1 at time t, and one birth occurs in the interval (t, t+h]; the associated conditional prob. is :-

$p_{K-1,K}(t, t+h) = q_{K-1,K}(t) \cdot h + o(h)$
$= \lambda_{K-1} \cdot h + o(h)$

3. The system is in state K+1 and one death occurs in the interval $(t, t+h]$; the associated conditional prob. is:

$p_{k+1,k}(t, t+h) = q_{k+1,k}(t) \cdot h + o(h)$

$= \mu_{k+1} \cdot h + o(h)$

4. Two or more transitions occur in the interval $(t, t+h]$, resulting in $X(t+h) = k$; the associated conditional prob. is:-

$o(h)$

Then, by theorem of total probability:-

$P[X(t+h) = k] = P_k(t+h)$

$= P_k(t) \cdot p_{k,k}(t, t+h) + P_{k-1}(t) \cdot p_{k-1,k}(t, t+h)$

$+ P_{k+1}(t) \cdot p_{k+1,k}(t, t+h) + o(h)$

→ After rearranging, dividing by h and taking limit as $h \to 0$, we get:-

$$\frac{dP_k(t)}{d(t)} = -(\lambda_k + \mu_k) P_k(t) + \lambda_{k-1} P_{k-1}(t) + \mu_{k+1} \cdot P_{k+1}(t) \quad , k \geq 1 \quad -(1)$$

Put $k = 0$

$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad -(1)$$

→ Put eq $\frac{dP_k(t)}{dt} = 0$ to get the derivative. The resulting set of difference equations provide the steady-state solution of Markov chain.

Let p_k denote the steady-state prob. that the chain is in state k

i.e. $p_k = \lim_{t \to \infty} P_k(t)$

→ Then,

$0 = -(\lambda_k + \mu_k) p_k + \lambda_{k-1} p_{k-1} + \mu_{k+1} \cdot p_{k+1} \quad , k \geq 1 \quad (2)$

$0 = -\lambda_0 p_0 + \mu_1 p_1 \quad -(3)$

The eq^ns (2) and (3) are also called as Balance equations.

→ By rearranging eq^n (2), we get

$\lambda_k p_k - \mu_{k+1} \cdot p_{k+1} = \lambda_{k-1} \cdot p_{k-1} - \mu_k \cdot p_k = \cdots = \lambda_0 p_0 - \mu_1 p_1$

But from eq^n (3), we have

$\lambda_0 p_0 - \mu_1 p_1 = 0$

then, $\lambda_{k-1} \cdot p_{k-1} - \mu_k p_k = 0$

And, hence $p_k = \frac{\lambda_{k-1}}{\mu_k} \cdot p_{k-1} \quad , k \geq 1$

$\therefore p_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \mu_2 \cdots \mu_k} p_0$

$= p_0 \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} \quad , k \geq 1$

Since, $\sum_{k \geq 0} p_k = 1$

We have:

$$\boxed{p_0 = \frac{1}{1 + \sum_{k \geq 1} \prod_{i=0}^{k-1} \left(\frac{\lambda_i}{\mu_{i+1}}\right)}}$$

→ Special Case of Birth-Death Processes:

1. Pure Birth Process:

If the death rate, $\mu_k = 0$ for all $k = 1, 2, \ldots$ then we have pure birth process.

Also, if we have constant birth rates i.e. $\lambda_k = \lambda (k = 0, 1, 2, \ldots)$, then we have the familiar poisson process.

$\therefore$ eqn ① becomes

$\frac{dP_0(t)}{dt} = -\lambda P_0(t)$ for $k = 0$

$\frac{dP_k(t)}{dt} = \lambda P_{k-1}(t) - \lambda P_k(t) \quad , k \geq 1$

where we have assumed that the initial state $N(0) = 0$, so that:

$P_0(0) = 1 \quad , \quad P_k(0) = 0$ for $k \geq 1$

2. Pure Death Process:

When birth rates are all assumed to be zero ie $\lambda_k = 0$ for all k, then we have Pure-Death process.
The Sm starts in some state $n \geq 0$ at time $t=0$ and eventually decays to state 0. Thus, state 0 is an absorbing state.

We consider 2 special cases of interest :-

i) Death Process with a constant rate:

In addition to $\lambda_i = 0$ for all i, we have $\mu_i = \mu$ for all i

$$\frac{dP_n(t)}{dt} = -\mu P_n(t), \quad k=n$$

$$\frac{dP_k(t)}{dt} = -\mu P_k(t) + \mu P_{k+1}(t), \quad 1 \leq k \leq n-1$$

$$\frac{dP_0(t)}{dt} = \mu P_1(t), \quad k=0$$

where we have assumed that the initial state $N(0)=n$, so that:
$P_n(0)=1$, $P_k(0)=0$, $0 \leq k \leq n-1$

ii) Death Process with linear rate.

When birth rates are all assumed to be zero ie $\lambda_i = 0$ for all i, and $\mu_i = i\mu$, $i = 1, 2, \ldots n$

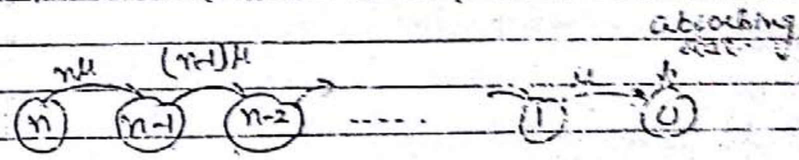
then,

$$\frac{dP_n(t)}{dt} = -n\mu P_n(t), \quad k=n$$

$$\frac{dP_k(t)}{dt} = -k\mu P_k(t) + (k+1)\mu P_{k+1}(t), \quad 1 \leq k \leq n-1$$

$$\frac{dP_0(t)}{dt} = \mu P_1(t), \quad k=0$$

where we have assumed that the initial state $N(0)=n$, so that:
$P_n(0)=1$, $P_k(0)=0$, $0 \leq k \leq n-1$

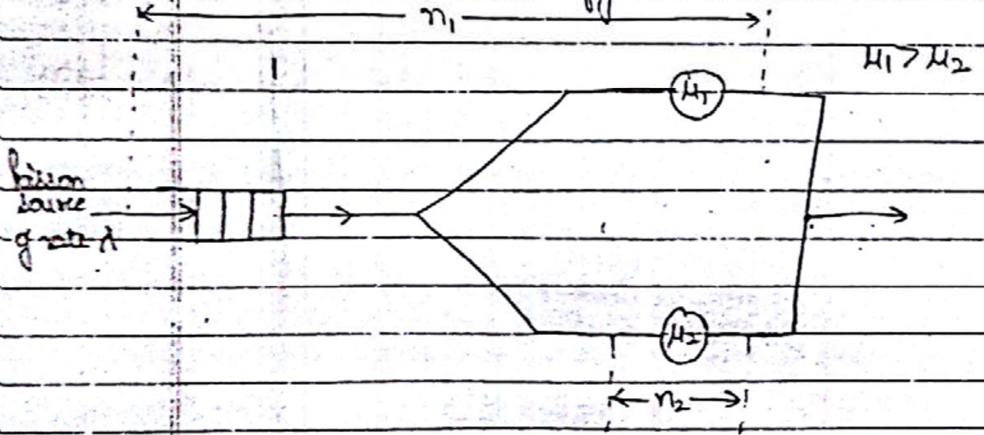


3. Non-Birth-Death Process:

A continuous parameter markov chain that do not satisfy the restriction of one-step transitions, such Markov chains are called Non-Birth-Death Processes.

Example: M/M/2 Queue with Heterogeneous Servers

Consider a variant of M/M/2 queue where the service rates of 2 processors are not identical. Such heterogeneous multiprocessor S/m is shown in fig. below :-



Assume without loss of generality that $\mu_1 > \mu_2$

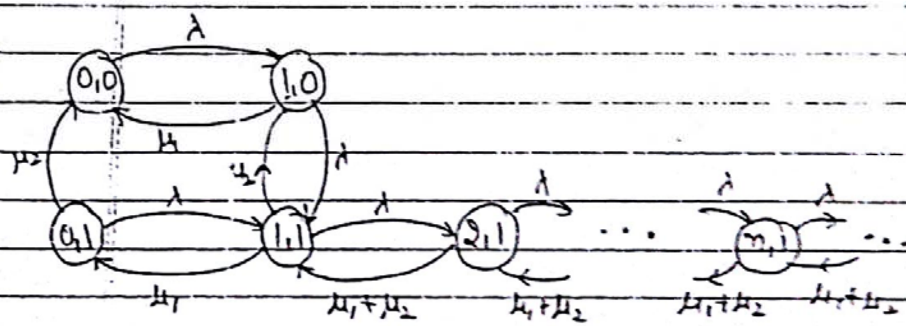
Let n_1 denotes the no. of jobs in the queue at the faster server, where $n_1 \geq 0$

Let n_2 denotes the no. of jobs in the queue at the slower server, and $n_2 \in \{0,1\}$.

The state of the S/m is defined to be the tuple (n_1, n_2)

Jobs wait in line in the order of their arrival.

State Diagram:



Balance equations in the steady state can be written by equating the rate of flow into a state to the rate of flow out of that state :-

$$\lambda p(0,0) = \mu_1 p(1,0) + \mu_2 p(0,1)$$

$$(\lambda+\mu_1)p(1,0) = \mu_2 p(1,1) + \lambda p(0,0)$$

$$(\lambda+\mu_2)p(0,1) = \mu_1 p(1,1)$$

$$(\lambda+\mu_1+\mu_2)p(1,1) = (\mu_1+\mu_2)p(2,1) + \lambda p(0,1) + \lambda p(1,0)$$

general,

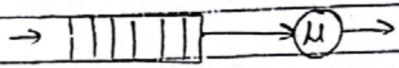
$$(\lambda+\mu_1+\mu_2)p(n,1) = (\mu_1+\mu_2)p(n+1,1) + \lambda p(n-1,1), \quad n \geq 1$$

→ Other Special Cases of Birth-Death Process'

1. M|M|1 Queue

Consider a single server Markov chain

Poisson arrival stream with average rate λ →



The job arrivals from a poisson process with mean rate λ.
The job interarrival times are exponentially distributed with mean $\frac{1}{\lambda}$

Similarly, the service time distributions are also exponentially distributed with mean $\frac{1}{\mu}$

- The scheduling discipline used is FCFS scheduling

Let $N(t)$ denotes the no. of jobs in the S/m at time t. Then $\{N(t), t \geq 0\}$ is a birth-death process with:

$$\lambda_k = \lambda, \quad k \geq 0$$

and,

$$\mu_k = \mu, \quad k \geq 1$$

SMCS . 4

$$p_Y(y) = \frac{1-F_X(y)}{E[X]}, \text{ we get}$$

$$\boxed{p_{T_i}(j) = \frac{1-F_i(j)}{E[X_i]}}$$

Now, for a window size τ, the probability that a page i is in memory (*i.e.*, in the working the time of random entry is given by

$$P(T_i < \tau) = \sum_{j=0}^{\tau-1} p_{T_i}(j)$$

$$\boxed{P(T_i < \tau) = \sum_{j=0}^{\tau-1} \frac{1-F_i(j)}{E[X_i]}} \quad \text{...(22) [Using equation}$$

Now, define a Bernoulli random variable Y_{it} = page i is in memory at time t. Then the exp value of Y_{it} is given as

$$\boxed{E[Y_{it}] = P(Y_{it} = I)}$$

which, in the limit, equal $P(T_i < \tau)$

Now the average working-set size is :

$$S(\tau) = \sum_{i=1}^{n} P(T_i < \tau)$$

$$\boxed{S(\tau) = \sum_{i=1}^{n} \sum_{j=0}^{\tau-1} \frac{1-F_i(j)}{E(X_i]}} \quad \text{...(24) [Using equation}$$

(*b*) **The average page-fault rate $q(\tau)$:** The conditional probability of a page fault give the i^{th} page is referenced at time t is $1 - F_i(\tau)$ as this is the probability that the inter reference of the page exceeds the window size; *i.e.*, the page has not been referenced during the last τ refere Therefore,

$$\boxed{P(\text{Page fault at time } t \mid r_t = i) = 1 - F_i(\tau)}$$

Now, using the theorem of total probability :

$$P(\text{Page fault at time } t) = \sum_{i=1}^{n} [1 - F_i(\tau)]\, P(r_t = i)$$

$$\boxed{P(\text{Page fault at time } t) = \sum_{i=1}^{n} [1 - F_i(\tau)]\, m_i(t)} \quad \text{...(26) [As } P(r_t = i) =$$

Now, taking the limit as t approaches infinity, the LHS is the asymptotic average-fault rate $q(\tau)$, and, using equation (17) we get :

$$\boxed{q(\tau)=\sum_{i=1}^{n} m_i[1-F_i(\tau)]=\sum_{i=1}^{n}\frac{1-F_i(\tau)}{E[X_i]}} \quad ...(27)$$

5.8 SOME COMMON EXAMPLES OF STOCHASTIC PROCESSES

In this section we discuss some common examples of stochastic processes. These are given as below :

(i) Consider a computer system with jobs arriving at random points in time, queuing for service, and departing from the system after service completion. Let N_k be the number of jobs in the system at the time of the departure of the k^{th} customer (after service completion). the stochastic process $\{N_k \mid k = 1, 2, ...\}$ is a **discrete-parameter, discrete-state process** with the state space $I = \{0, 1, 2, ...\}$ and the index-set $T = \{1, 2, 3.......\}$

(ii) Let $X(t)$ be the number of jobs in the system at time t. then $\{X(t) \mid t \in T\}$ is a **continuous-parameter, discrete-state process** with $I = \{0, 1, 2,......\}$ and $T = \{t \mid 0 \le t < \infty\}$.

(iii) Let W_k be the time that the k^{th} customer has to wait in the system before receiving service. Then $\{W_k \mid k \in T\}$, with $I = \{x \mid 0 \le x < \infty\}$ and $T = \{1, 2, 3, ...\}$ is a **discrete-parameter, continuous-state process.**

(iv) Let $Y(t)$ denote the cumulative service requirement of all jobs in the system at time t. Then $\{Y(t) \mid 0 \le t < \infty\}$ is a **continuous-parameter, continuous-state** process with $I = [0, \infty)$.

The daily life examples that can be modeled as stochastic processes includes

(a) The scores of consecutive NFL games.

(b) The daily prices of a stock.

(c) The thermal noise in a resistor.

(d) The numbers generated by successive spinning of a roulette wheel.

(e) The winning or losses of a roulette player.

279 Ques -1

EXAMPLE 5.1 Show that the time that a discrete-parameter homogenous markov chain spends in a given state has a geometric distribution.

Solution. For a homogenous discrete-parameter markov chain we have

$$P(X_{n+1} = j \mid X_n = i) = p_{ij}$$

independent of the past history and independent of n.

Then consider the event $A = [X_{n+1} \ne i$ given that $X_n = i]$. We call the occurrence of this event as a success.

Therefore $P(A) = 1 - p_{ii} = \sum_{j \ne i} p_{ij}$

Thus, whenever $X_n = i$, we perform a Bernoulli trial until a success occurs making $X_{n+k} \ne i$. The random variable Y_i denoting the holding time then has the pmf.

P_{ii} = Prob. to remain on state i

$(1 - P_{ii})$ to change the state

$P_{ii}(Y_i = k)\ (1 - P_{ii}) P_{ii}^{\,k-1}$

$$p_{Y_i}(k) = P(Y_i = k) = P(X_{n+k} \neq i,\ X_{n+k-1} = i,\ \ldots\ldots,\ X_{n+1} = i \mid X_n = i)$$

Now using the **multiplication rule** we get

$$p_{Y_i}(k) = P(X_{n+k} \neq i,\ X_{n+k-1} = i, \ldots, X_{n+2} = i \mid X_{n+1} = i, X_n = i).\ P(X_{n+1} = i \mid X_n = i)$$

$$p_{Y_i}(k) = P(X_{n+k} \neq i \mid X_{n+k-1} = i, \ldots X_n = i) \ldots P(X_{n+1} = i \mid X_n = i)$$

$$p_{Y_i}(k) = P(X_{n+k} \neq i \mid X_{n+k-1} = i) \ldots P(X_{n+1} = i \mid X_n = i)$$

$$\boxed{p_{Y_i}(k) = (1 - p_{ii})\, p_{ii}^{\,k-1}}$$

Clearly, Y_i has a geometric distribution with parameter $(1 - p_{ii})$.

EXAMPLE 5.2. Show that the time that a homogenous, continuous-parameter Markov chain spends in a given state has an exponential distribution.

Solution. The (time) homogenous Markov chain has the property of invariance with respect to the time origin t_n *i.e.*,

$$P[X(t) \leq x \mid X(t_n) = x_n] = P[X(t - t_n) \leq x \mid X(0) = x_n]$$

For a homogenous Markov chain, the past history of the process is completely summarized in the current state. Therefore, the distribution for the time Y the process spends in a given state must be memoryless. *i.e.*,

$$P(Y \leq t + t \mid Y \geq t) = P(Y \leq r)$$

i.e., $$P(Y \leq\ ) = \frac{P(t \leq Y \leq t + r)}{P(Y \geq t)}$$

$\Rightarrow$ $$F_Y(r) = \frac{F_Y(t + r) - F_Y(t)}{1 - F_Y(t)}$$

Now, Dividing both sides by r and taking the limit as r approaches zero, we get

$$F_Y'(0) = \frac{F_Y'(t)}{1 - F_Y(t)}$$

The above differential equation has the unique solution.

$$\boxed{F_Y(t) = 1 - e^{-F_Y'(0)t}}$$

Clearly, the homogenous, continuous-parameter Markov chain spends in a given state has an exponential distribution.

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EXAMPLE 5.3 Show that the autocorrelation function R (t_1, t_2) of a strict-sense stationary process depends only on the time difference $(t_2 - t_1)$ if it exists.

Solution. According to the definition of strict-sense stationary process, we get

$$F_{X(t_1),X(t_2)}(x_1, x_2; t_1, t_2) = F_{X(t_1+\tau),X(t_2+\tau)}(x_1, x_2; t_1 + \tau, t_2 + \tau)$$

$$= E[x[(t_1+\tau), (t+\tau))]]$$

$$= \iint x_1 x_2 f(x_1, x_2 \cdot t_1 + \tau, t_2 + \tau)$$

[As $F(x;t) = F(x;t+\tau)$ According to the definition of strict-sense stationary process]

Now assume that $X(t)$ has a continuous state space

$$f_{X_1(t_1),X(t_2)}(x_1,x_2;t_1,t_2) = f_{X(t_1+\tau);X(t_2+\tau)}(x_1,x_2;t_1+\tau,t_2+\tau)$$

Then, the autocorrelation function

$$R(t_1,t_2) = E[X(t_1)\,X(t_2)] = \iint x_1 x_2 \underline{f_{X(t_1),X(t_2)}(x_1,x_2;t_1,t_2)}\,dx_1\,dx_2 = E[X[(t_1+\tau),(t_2+\tau)]$$

$$= R(t_1+\tau,\ t_2+\tau) \qquad = \iint x_1 x_2 f(x_1,x_2;t_1+\tau,t_2+\tau)$$

Hence the proof. 283 Ques-1.

EXAMPLE 5.4 **Show that the first order interarrival time of the non-homogeneous Bernoulli process is not memoryless.**

Solution. For a non-homogeneous Bernoulli process, we have

$$P(Y_1 = i) = (1-p_1)(1-p_2)\ldots\ldots(1-p_{i-1})p_i$$

$$P(Y_1 = i+j \mid Y_1 > j) = \frac{P(Y_1 = i+j)}{P(Y_1 > j)} = \frac{P(Y_1 = i+j \;\&\; Y_1 > j)}{P(Y_1 > j)} = \frac{P(Y_1 = i+j)}{P(Y_1 > j)}$$

$$P(Y_1 = i+j \mid Y_1 > j) = \frac{(1-p_1)(1-p_2)\ldots\ldots(1-p_{i+j-1})p_{i+j}}{\sum_{i=j+1}^{\infty}\prod_{k=i}^{i-1}(1-p_k)\,p_i}$$

$$P(Y_1 = 2 \mid Y_1 > 1) = \frac{(1-p_1)p_2}{(1-p(Y_1=1))} = \frac{(1-p_1)p_2}{1-p_1} = p_2$$

But $P(Y_1 = 1) = p_1 \neq p_2$. Hence the required result.

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EXAMPLE 5.5. **Consider a stochastic process defined on a finite sample space with three sample points. Its description is provided by the specifications of the three sample functions:**

$X(t,s_1) = 3,\ X(t,s_2) = 3\cos(t) \quad X(t,s_3) = 4\sin(t).$ $\qquad E[X] = 0\cdot P_X(0) + 1\cdot P_X(2) + 2\cdot P_X(2)$

Also given is the probability assignment formula

$P(s_1) = P(s_2) = P(s_3) = 1/3.$

Compute $\mu(t) = E[X(t)]$ and the autocorrelation function $R(t_1,t_2)$. Now answer the following questions. Is the process strict-sense stationary ? Is it wide-sense stationary ?

Solution. $\mu(t) = \left(3\times\frac{1}{3} + 3\cos(t)\times\frac{1}{3} + 4\sin(t)\times\frac{1}{3}\right)$ $\qquad E[XY] = \sum xy\,P(xy)$

$$\boxed{\mu(t) = 1 + \cos(t) + \frac{4}{3}\sin(t)} \qquad \ldots(i) \qquad xy\,P(X=x, Y=y)$$

$R(t_1,t_2) = E(X(t_1)\,X(t_2))$

$R(t_1,t_2) = 9\,P(X(t_1) = 3, X(t_2) = 3) + 9\cos(t_1)\cos(t_2)\,P(X(t_1) = 3\cos(t_1), X(t_2) = 3\cos(t_2)) + 16\sin(t_1)\sin(t_2)\,P(X(t_1) = 4\sin(t_1).\,X(t_2) = 4\sin(t_2))$

$$R(t_1, t_2) = 9 \times \frac{1}{3} + 9\cos(t_1)\cos(t_2) \times \frac{1}{3} + 16\sin(t_1)\sin(t_2) \times \frac{1}{3}$$

$$\boxed{R(t_1, t_2) = 3 + 3\cos(t_1)\cos(t_2) + \frac{16}{3}\sin(t_1)\sin(t_2)} \qquad \ldots(2)$$

Now, for a process to be strict-sense stationary, $\mu(t) = \mu$ for all $t \in T$. But Here is not true as clear from equation (1).

Therefore the process is not strict-sense stationary process.

Also, for a process to wide-sense stationary process we must have

(*i*) $\mu(t) = E[X(t)]$ is independent of t.

(*ii*) $R(t_1, t_2) = R(0, t_2 - t_1) = R(\tau),\ t_2 \geq t_1 \geq 0$

(*iii*) $R(0) = E[X^2(t)] < \infty$ [finite second moment]

As from equation (1) and (2) if is clear that no condition is being satisfied for this given process.

Clearly the given process is not a wide-sense stationary process.

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EXAMPLE 5.6. Assuming that the number of arrivals in the interval (0, t] is Poisson distribution with parameter λt. Compute the probability of an even number of arrivals. Also, compute the probability of an even number of arrivals. Also, compute the probability of an odd number of arrivals.

Solution. Let the number of arrivals be denoted by N_t. Clearly, N_t is poisson distributed with parameter λt [As mentioned in the problem]. Therefore, the required probability is

$$P(N_t = \text{even}) = \sum_{j=0}^{\infty} P(N_t = 2j)$$

$2j = i$

$$= e^{-\lambda t}\sum_{j=0}^{\infty}\frac{(\lambda t)^{2j}}{(2j)!} = \frac{1}{2}e^{-\lambda t}\sum_{j=0}^{\infty}\frac{2(\lambda t)^{2j}}{(2j)!}$$

$\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$

$$= \frac{1}{2}e^{-\lambda t}\left[\sum_{i=0}^{\infty}\frac{(\lambda t)^i}{i!} + \sum_{i=0}^{\infty}(-1)^i\frac{(\lambda t)^i}{i!}\right] = \frac{1}{2}e^{-\lambda t}[e^{\lambda t} + e^{-\lambda t}]$$

$$= \frac{1 + e^{-2\lambda t}}{2}, \qquad t \geq 0$$

Similarly, $P(N_t = \text{odd}) = \dfrac{1 - e^{-2\lambda t}}{2},\ t \geq 0.$

EXAMPLE 5.7. Consider a file-management-system that employs blocking of several logical records on a single physical block. Now if records are accessed sequentially, a logical I/O will require a physical I/O operation only occasionally. Assume that the probability that a logical I/O will generate a

known a physical to be 0.75. We are integrated in studying the usage profile of a profile of a specific file F. Assume that the probability that given I/O operation refers to a record of file F is 2/3. Further assume that the two events "File F requested" and "logical I/O give rise to physical I/O are independent and that successive I/O operations are independent.

(a) Draw the tree diagram for the above process.

(b) Show that the above sequence of logical I/O operations can be modeled as a Bernoulli process.

Solution (*a*) The tree diagram is as shown in figure 5.6.

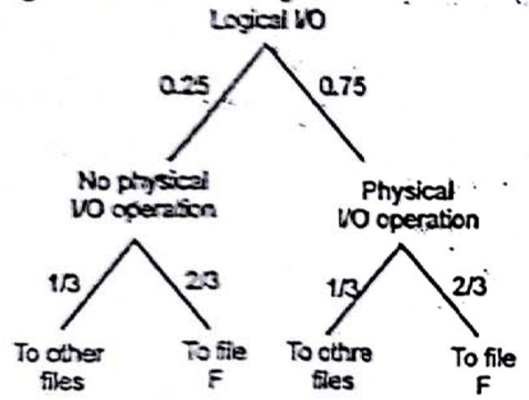

Fig. 5.6 The diagram for accessing a file.

(*b*) From the tree diagram shown in figure 5.6 it is clear that we may consider the sequence of logical I/O operations a Bernoulli process with probability of success (a logical I/O course physical I/O to file F) equals to $\frac{3}{4} \times \frac{2}{3} = 0.5$. Note here that the number of logical I/O operations between two physical I/O operations directed to file F will then be geometrically distributed with parameter $p = 0.5$.

EXAMPLE 5.8. Suppose that there are two independent Poisson job-arrival streams into a computing centre, with respective arrival rates λ_1 and λ_2. For the pooled job-arrival stream show the arrival times of job stream 1, of job stream 2, and of the pooled stream.

Solution. The resultant pooled-job arrival stream is as shown below in figure 5.7.

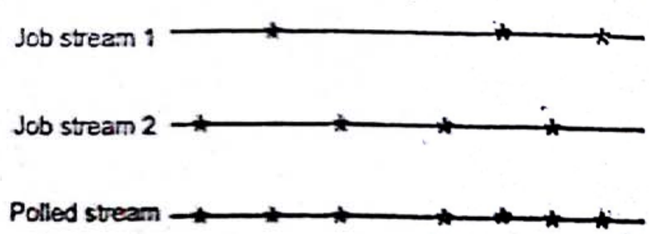

Fig. 5.7 Pooling two poisson streams.

In P^2, the condition of irreducible Markov chain is being satisfied. There for $n = 2$ the chain in ...cible.

For calculating steady stated probabilities

$$v = vP$$

$$\Rightarrow \quad (v_0, v_1, v_2) = (v_0, v_1, v_2)\begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{bmatrix}$$

$$v_0 = 0.6 + 0.1\, v_1 + 0.6\, v_2$$
$$v_1 = 0.2\, v_0 + 0.8\, v_1 + 0.0\, v_2$$
$$v_2 = 0.2\, v_0 + 0.1\, v_1 + 0.4\, v_2$$

Also solving these equation we get

$$v_0 = \frac{2}{5}, \ v_1 = \frac{2}{5} \text{ and } v_2 = \frac{1}{5}$$

$\therefore$ Steady state probability vector $v = \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$.

EXAMPLE 6.10. Consider a two-state Markov chain with the condition $0 < a, b < 1$ and $|1 - a - b| < 1$ such that

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}.$$

Find the steady state probability vector.

Solution. Using $v = vP$

$$\Rightarrow \quad (v_0, v_1) = (v_0, v_1)\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

$$v_0 = (1-a)\, v_0 + bv_1$$
$$v_1 = av_0 + (1-b)\, v_1$$

and $\quad v_0 + v_1 = 1$

On solving these equations we get

$$v_0 = \frac{b}{a+b}, \quad v_1 = \frac{a}{a+b}$$

Therefore, the steady state vector is $v = (v_0, v_1) = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)$.

Alternatively, By using $\lim\limits_{n\to\infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix} = \begin{bmatrix} v_0 & v_1 \\ v_0 & v_1 \end{bmatrix}$

Therefore, the steady state vector is $v = (v_0, v_1) = \left(\frac{b}{a+b}, \frac{a}{a+b}\right)$.

EXAMPLE 6.11 Consider a system with two components. We observe the stat of the system every hour. A given component operating at time n has probability p of failing before the next observation at time $n+1$. A component that was in a failed condition at time n has a probability r of being repaired by time $n+1$, independent of how long the component has been in a failed state. The component failures and repairs are mutually independent events. Let X_n be the number of components in operation at time n, $\{X_n \mid n = 0, 1, \ldots\}$ is a discrete-parameter homogenous Markov chain with the state space $I=\{0, 1, 2\}$. Determine its transition probability matrix P, and draw the state diagram. Obtain the steady-state probability vector, if it exists.

Solution. Consider the state diagram shown in figure 6.18

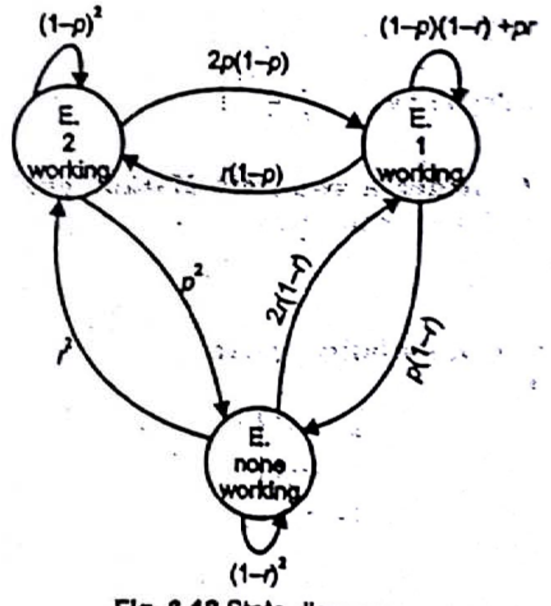

Fig. 6.18 State diagram.

From the state diagram of figure 6.18 the one step transition probability matrix can be written as :

$$P = \begin{bmatrix} (1-r)^2 & 2r(1-r) & r^2 \\ p(1-r) & 1-r-p & r(1-p) \\ p^2 & 2p(1-p) & (1-p^2) \end{bmatrix}$$

Now, let the steady state vector $v = (v_0, v_1, v_2)$

$$\Rightarrow \quad (v_0, v_1, v_2) = (v_0, v_1, v_2)\begin{bmatrix} (1-r)^2 & 2r(1-r) & r^2 \\ p(1-r) & 1-r-p & r(1-p) \\ p^2 & 2p(1-p) & (1-p)^2 \end{bmatrix}$$

$$\Rightarrow \quad v_0 = (1-r)^2 v_0 + p(1-r)v_1 + p^2 v_2$$
$$v_1 = 2r(1-r)v_0 + (1-r-p)v_1 + 2p(1-p)v_2$$
$$v_2 = r^2 v_0 + r(1-p)v_1 + (1-p)^2 v_2$$

Also $\quad v_0 + v_1 + v_2 = 1$

On solving these equations we get

$$v_0 = \frac{p^2}{(p+r)^2},\ v_1 = \frac{2pr}{(p+r)^2},\ v_2 = \frac{r^2}{(p+r)^2}$$

Therefore, the steady state vector $v = (v_0, v_1, v_2) = \left(\frac{p^2}{(p+r)^2}, \frac{2pr}{(p+r)^2}, \frac{r^2}{(p+r)^2}\right)$.

EXAMPLE 6.12. Any transition probability matrix P is a stochastic atrix, that is, $p_{ij} \geq 0$ for all i and all j and $\sum_j p_{ij} = 1$, for all i. If in additions he column sums are also unity that is $\sum_i p_{ij} = 1$, for all j, then matrix P is alled doubly stochastic. If a Markov chain with doubly Stochastic P is reducible, aperiodic and finite with n states, show that the steady-state robability is given by

$$v_j = \frac{1}{n}, \text{ for all } j.$$

olution. Using the equation

$$v = vP$$

$$\Rightarrow \quad v_j = \sum_i v_i p_{ij}$$

Certainly, $v_j = \frac{1}{n}$ is a solution since $\sum_i p_{ij} = 1$. Now if the discrete time Markov chain DTMC) satisfies the given properties, then since $\sum_j v_j = \frac{1}{n} = 1$, the above solution must be the nique steady-state probability vector.

EXAMPLE 6.13. Consider a model of a uniprogrammed computer, ystem with m I/O devices and a CPU. For the program currently under xecution, the system will be in one of the $m+1$ states denoted by 0, 1, 2,........, o that in state 0 the program is executing on the CPU, and in state i ($1 \leq i \leq m$) the program is performing an I/O operation on device. Assume that the request for device i occurs at the end of a CPU burst with probability q_i, independent of the past history of the program. The program will finish

Therefore, the Fundamental Matrix M is computed to be

$$M = (1 - Q)^{-1} = \begin{bmatrix} 1 & 0.7791 & 0.8753 & 0.8254 \\ 0 & 1.105 & 0.5236 & 0.8721 \\ 0 & 0.2907 & 0.453 & 0.7558 \\ 0 & 0.1744 & 0.8721 & 1.453 \end{bmatrix}$$

Thus the vertices s_1, s_2, s_3 and s_4 are executed 1, 0.7791, 0.8953, and 0.8254 times on the average. The average execution time of the program is equal to :

$$\tau = t_1 + 0.7791\, t_2 + 0.8953\, t_3 + 0.8254\, t_4 + t_5 \text{ time units.}$$

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EXAMPLE 6.26 **Given the stochastic program flow graph shown in Figure 6.32 compute the average number of times each vertex s_i is visited, and assuming that the execution time of s_i is given by $t_i = 2i + 1$ (i = 1, 2, 3, 4, 5, 6) time units, find the average total execution τ of the program.**

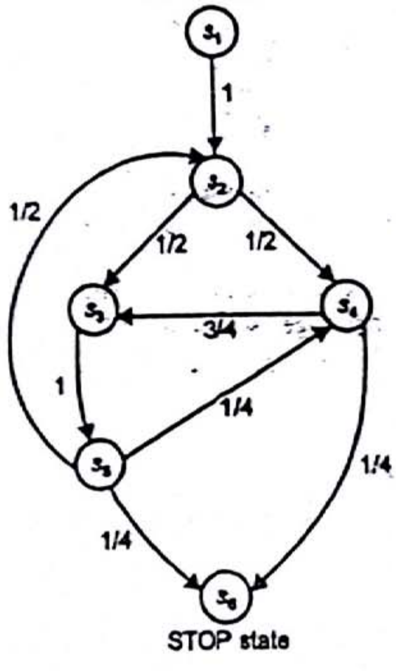

Fig. 6.32 A program flow graph.

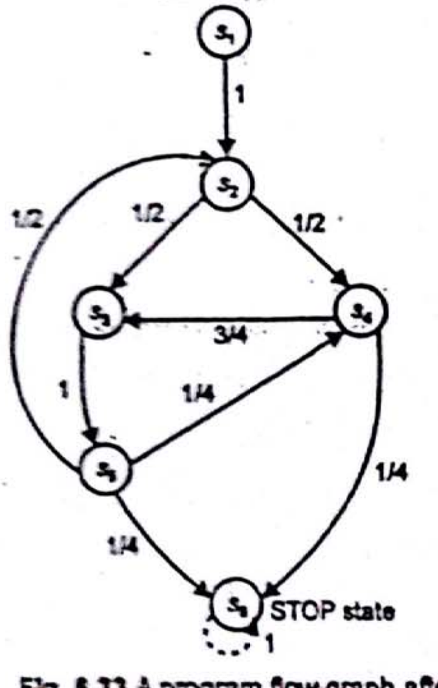

Fig. 6.33 A program flow graph after adding a dummy edge on STOP state s_6.

Solution. Here s_6 is a 'final' or 'stopping' state and hence is an absorbing state and the remaining states are transient states. Therefore, we introduce a dummy edge (self-loop) of weight 1 on s_6 represented by dotted edge as shown in figure 6.33. Therefore, the transition probability matrix P is given by

$$P = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{array} \begin{array}{c} \begin{array}{cccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \end{array} \\ \left[\begin{array}{ccccc:c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3/4 & 0 & 0 & 1/4 \\ 0 & 1/2 & 0 & 1/4 & 0 & 1/4 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

By partitioning P according to step (*vi*) mentioned in section 6.9.1 we get

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/4 & 0 \end{bmatrix}$$

Therefore, the Fundamental matrix M is computed to be

$$M = (1-Q)^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -3/4 & 1 & 0 \\ 0 & -1/2 & 0 & -1/4 & 1 \end{bmatrix}$$

$$\Rightarrow \quad M = \begin{bmatrix} 1 & 13/6 & 7/3 & 5/3 & 7/3 \\ 0 & 13/6 & 7/3 & 5/3 & 7/3 \\ 0 & 4/3 & 8/3 & 4/3 & 8/3 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 4/3 & 5/3 & 4/3 & 8/3 \end{bmatrix}$$

Thus the vertices s_1, s_2, s_3, s_4 and s_5 are executed 1, $\frac{13}{6}, \frac{7}{3}, \frac{5}{3}$ and $\frac{7}{3}$ times on the average. Also given that $t_i = 2i + 1$, $(i = 1, 2, 3, 4, 5, 6)$, therefore, the average execution time of the program is equal to :

$$\tau = 1(3) + \frac{13}{6}(5) + \frac{7}{3}(7) + \frac{5}{3}(9) + \frac{7}{3}(11) + 13 \text{ time units}$$

$$\tau = 83 + \frac{5}{6} \text{ time units}$$

$$\tau = \frac{493}{6} \text{ time units}$$

EXAMPLE 5.9. **There are n independent sources of environmental shocks to a component. The number of shocks from the i^{th} source in the interval $(0, t]$, denoted $N_i(t)$, is governed by a Poisson processes with rate λ_i. Find the rate λ for the total number of shocks of all kinds in the interval $(0, t]$.**

Solution. According to the superposition of independent Poisson processes, we conclude that the total number of shocks of all kinds in the interval $(0, t]$ forms a Poisson process with the rate.

$$\boxed{\lambda = \lambda_1 + \lambda_2 + \ldots\ldots + \lambda_n}$$

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EXAMPLE 5.10 **Consider a computer system with Poisson job-arrival stream at an average rate of 60 per hour. Determine the probability that the time interval between successive job arrivals is :**

(a) Longer than four minutes.

(b) Shorter than eight minutes.

(c) Between two and six minutes.

Solution. (a) $e^{-4} = 0.0183$ $\Rightarrow P(X>4) = 1-F_X(4) = 1-1+e^{-1\times 4}$

(b) $1 - e^{-8} = 0.999665$ $b = P(X \le 8) = F_X(8) = 1-e^{8}$

(c) $e^{-2} - e^{-6} = 0.13286.$ $c = P(2 < X < 6) = e^{-2} - e^{-6}$

EXAMPLE 5.11. **Consider the generalization of the ordinary Poisson process, called the compound Poisson process. In an ordinary Poisson process, we assumed that the probability of occurrence of multiple events in a small interval is negligible with respect to the length of the interval. If the arrival of a message in a computer communication system is being modeled, the counting process may represent the number of bytes (or packets) in a message. In this case suppose that the pmf of the number of bytes in a message is specified.**

P(number of bytes in a message = k) = a_k, $k \ge 1$. Further assume that the number of message arrivals from an ordinary Poisson process with rate λ. The process $\{X(t) \mid t \ge 0\}$, where $X(t)$ = number of bytes in the interval $(0, t]$ is a compound Poisson process. Show that the generating function of $X(t)$ is given by :

$$G_{X(t)}(z) = e^{\lambda t[G_A(z)-1]}$$

where $$G_A(z) = \sum_{k \ge 1} a_k z^k .$$

Solution. $X(t) = \sum_{i=1}^{N(t)} A_i$ where A_i is the number of bytes in the i^{th} message and $N(t)$ is the number of message arrivals in the interval $(0, t)$. Let the PGF iof A_i be

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EXAMPLE 6.9 **Assume that a computer system is in one of three states : busy, idle, or undergoing repair, respectively denoted by 0, 1 and 2. Observing its state at 2 P.M. each day, we believe that the system approximately behaves like a homogenous Markov chain with the transition probability matrix.**

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{bmatrix}$$

Prove that the chain is irreducible and determine the steady state probabilities.

Solution. The state diagram is as shown in Figure 6.16

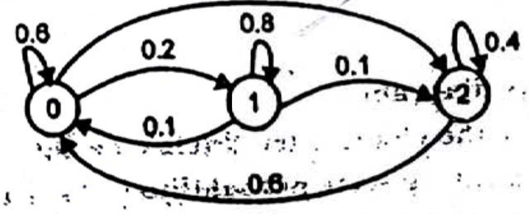

Fig. 6.16 State diagram for a three state computer system.

From the state diagram it is cleat that for $n = 1$ the property of irreducible Markov chain does not hold good.

For $n = 2$ we have

$$P^2 = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{bmatrix} \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.1 & 0.8 & 0.1 \\ 0.6 & 0.0 & 0.4 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.50 & 0.28 & 0.22 \\ 0.20 & 0.66 & 0.14 \\ 0.60 & 0.12 & 0.28 \end{bmatrix}$$

As all the entries are non-zero therefore the new state diagram in shown in figure 6.17.

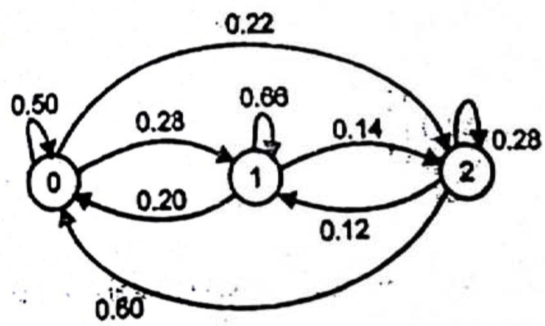

Fig. 6.17 State diagram for P^2.