(1)

UNIT-3 (Stochastic Process)

Let X is a single valued function then
$X(t)$ is the collection of values of a function
ie $X(t_1), X(t_2) \dots X(t_n)$

Similarly; Y is a single valued function then
Y_n is the collection of values
ie $Y_1\ Y_2 \dots Y_n$

⇒ Stochastic Process is defined as family of Random Variables
ie indexed by a parameter such as time is known
ie $S.P = [X(t,s) \mid t \in T, s \in S]$
Sample pt

⇒ Also defined as,
On a given probability space T; indexed by parameter t
where t varies over index set (T) and the values assumed by $X(t)$ are called states
And The set of all possible values forms the state space of process

⇒ For a fixed $t = t_1$, $X(t_1, s) = X_{t_1}(s)$ is a R.V denoted by
as s varies over sample space S
ie $X(t_1, s) \mid s \in S$
is called a Random function Denoted by $X(t_1) = s \mid s \in$

⇒ For a fixed state $s = s_1$ ie $X(t, s_1) \mid t \in T$ is called
a sample function & is denoted by $X(t) = s_1 \mid t \in$

Stochastic process
☆ If Both the sample pt s & time t varies
ie $X(t, s) \mid t \in T$ & $s \in S$
Then we have family of RV constituting stochastic process

Types of Stochastic Process

I) Depending on the Nature of State space S and Index Set T, It can be classified as

a) Discrete-state process
b) Continuous state process
c) Discrete-Parameter process
d) Continuous-Parameter process

A) <u>Discrete- State Process</u> ⇒ If the state space of stochastic process is discrete, then it is called discrete state process often referred to as chain

B) <u>Continuous State Process</u> ⇒ If the state space of Stochastic process is continuous then it is called Continuous state process

c) <u>Discrete (time-parameter process)</u> ⇒ If the Index set (T) of Stochastic process is discrete then it is called Discrete (time) parameter process

d) <u>Continuous parameter process</u> ⇒ If the Index set (T) of stochastic process is continuous then we have Continuous parameter process

II Depending on Distribution f^n & Density f^n, Stochastic process can be classified as

a) First Order function
b) Second Order function
c) N^{th} Order function

A) <u>First Order Function</u>

→ For a fixed time instance $t = t_1$, $X(t_1)$ is a simple R.V. that describes the value of state of process at time t_1

→ ~~For a fixed X_1, the probability of~~

→ For a fixed state $x = x_1$,

The CDF of stochastic process is defined as $F(x_1, t_1)$

* The probability of Event $X(t_1) \le x_1$ i.e. $P[X(t_1) \le x_1]$
* The CDF of stochastic process is $F(x_1, t_1) = F_{X(x_1)}(t_1) = P[X(t_1) \ldots$
* The Density f^n is defined as $f(x_1, t_1) = \frac{d}{dx} F_X(x_1, t_1)$

B) <u>Second Order function</u>

→ For a fixed time instance $t = t_1, t_2$
and for a fixed value $X = x_1\ x_2$

Distribution of stochastic process is defined as

* The CDF of Stochastic process is $F_X(x_1, x_2, t_1 t_2) = P[X(t_1) \le x_1, X(t_2) \le x_2]$

* The Density f^n: $f_X(x_1, x_2; t_1, t_2) = \frac{d^2}{dx_1 dx_2} F_X(x_1, x_2; t_1, t_2)$

C) <u>N^{th} Order function</u>

For Index set $T = t_1\ t_2 \ldots t_n$
& state space $X = x_1\ x_2 \ldots x_n$

The Distribution f^n of n^{th} order stochastic process is given

* The CDF of S.P is $F(x_1\ x_2 \ldots x_n, t_1\ t_2 \ldots t_n) = F(x \in S, t \in T)$
$= P(X(t_1) \le x_1, X(t_2) \le x_2 \ldots X(t_n) \le x_n]$

* The Density f^n is given as

$$f(x_1 x_2 \cdots x_n;\ t_1 t_2 \cdots t_n) = f(x \in S;\ t \in T)$$
$$= \frac{d^n F_X(x \in S\ \ t \in T)}{dx_1 dx_2 \cdots dx_n} = \frac{d^n F(x_1 x_2 \cdots x_n,\ t_1 t_2 \cdots t_n)}{dx_1 dx_2 \cdots dx_n}$$

Classification of Stochastic Process

Stochastic process can be classified as

(i) Strictly Stationary Process
(ii) Independent Process
(iii) Renewal Process
(iv) Markov Process
(v) Poisson Process
(vi) wide sense stationary Process
(vii) Bernoulli Process

$x \in I^n$
$t \in T^n$

I) STRICTLY STATIONARY PROCESS

A Stochastic Process is a strict sense stationary process if for all $n \geq 1$ i.e. $\forall n \geq 1$

$$F(x;t) = F(x, t+\tau) \quad \left[\begin{array}{l} \forall x \in R^n \\ \& t \in T^n \end{array}\right]$$

we let $\mu(t) = E[x(t)]$ and is called Expectation of the process $[x(t)]$

Applying the definition of strict sense to 1st Order CDF we get

$$F(x,t) = F(x, t+\tau)$$

i.e. $F_X(t) = F_X(t+\tau)$

Applying this Def to n^{th} order we get

$$F(x_1\, x_2 \dots x_n,\ t_1+\tau,\ t_2+\tau\ t_3+\tau \dots t_n+\tau)$$
$$= F(x_1\, x_2 \dots x_n,\ t_1\ t_2 \dots t_n)$$

It follows that strict sense stationary process is time independent, where τ is a scalar quantity

* NOTE - Strict sense stationary process has time independent mean ie $\mu(t) = \mu \quad \forall\, t \in T$

∴ Mean is same for all values of t

II) INDEPENDENT PROCESS

A stochastic process $[X(t) / t \in T]$ is said to be Independent process provided its n^{th} order joint distribution satisfies the condition

$$F(\underline{x};\underline{t}) = \prod_{i=1}^{n} F(x_i, t_i).$$

$$= \prod_{i=1}^{n} P[X(t_i) \le x_i]$$

III) RENEWAL PROCESS

Definition : A Renewal process is defined to be a Discrete-parameter independent process $\{X_n / n = 1, 2 \dots\}$ where $X_1\, X_2 \dots$ are independent, identically distributed non-negative Random Variables

Renewal Process means; The system in which the repair/replacement after failure is performed

& time required in repairing / replacing is renewal time

for eg when bulb fuses then we change it with new bulb

Time taken to change the new bulb is > 0

... n is > 1

Consider $x_1, x_2, \ldots, x_n$ be sequence of non negative, Identically Distributed & Mutually Independent R.V. Then $X_i / i = 1, 2, \ldots n$ is called Renewal Process

→ As an Example of such a process, Consider the system with unlimited supply of new and similar light weight Bulbs
→ If one of them is selected, Inserted in a socket & allowed to burn until it fails,
→ It is then immediately replaced with other bulb. Then we can see dat

* Life of Bulb is Non Negative.
* Life of Bulb has same Distribution (Identically Distributed)
* Life of ~~Bulb~~ Bulb is ~~a~~ independent to other Bulbs


All 7 bulbs follows above 3 properties

So It can be treated as an Example of Renewal process

IV. $\boxed{\text{MARKOV PROCESS}}$

A stochastic process $X(t, s) / t \in T, s \in S$ is known to have markov process, if the probability Distribution for its future value depends only on the present state, but not on how the process is arrived at dat state.
ie Acc to Markov process, future value depends only on the current value but not on past value

for Example, Consider a Markov process for time
$t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n < t$

Then the conditional distribution of $X(t)$ depends only on $X(t_n)$

ie $P[X(t) \le x \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \dots X(t_0) = x_0]$
$= P[X(t) \le x \mid X(t_n) = x_n]$

For Example, In Recruitment process, any company depends only on present degree of student rather on his school time record

ie class 1, 2, 3, 4, 5, ..8, 10, 12 | U.G 1 2 3 4 | Future Organiser

future depend on, rather on past

$X(t) \le x$ [future],
$X(t_n) = x_n$ [present]
& $X(t_{n-1}) = x_{n-1}$, $X(t_{n-2}) = x_{n-2}$ ---- $X(t_0) = x_0$ [past]

5. $\boxed{\text{WIDE STATIONARY PROCESS}}$

Wide sense stationary process is strictly stationary process (But only for 2 values)
like t_1 & t_2 can adjust their time while rest remain ...

A stochastic process is known to have wide sense stationary if it depends on the non scalar f^n & if we apply the definition of strictly stationary process, then we can say that for a fixed time interval $t_1 t_2$ the values $x_1 x_2$ does not change with scalar quantity

ie $F(t_1+\tau, t_2+\tau; x_1 x_2) = F(t_1 t_2, x_1 x_2)$

* Density function of x_1, x_2 is $f_{x_1 x_2}(x_1 x_2)$

* Expectation of 2 Variable is $E[X(t_1) X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{x_1 x_2}(x_1 \dots$

$\left[\because Ex = \int_{-\infty}^{\infty} x \cdot \text{density } f^n \, dx\right]$

$X\{0,1,2\}$ $p\,X(0)=$ $p\,X(1)=$ $p\,X(2)$.

The Above function is also termed as Auto Correlation function
Now; Acc to Auto-Corelation function,
A stochastic process will be wide sense stationary when

$$E[X(t_1+\tau)\,X(t_2+\tau)] = E[X(t_1)\,X(t_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 x_2)\, f_{X_1X_2}(x_1 x_2;\, t_1+\tau\;\; t_2+\tau)\,dx_1 dx_2$$

Conclusion
A stochastic process will be W.S.S, iff its IInd order distribution is strictly stationary
and its IInd order Density f^n is strictly stationary.

vi) $\boxed{\text{Bernoulli Process}}$

It is a discrete state discrete parameter stochastic process consisting of a sequence of Independent r.v taking value 0 & 1
& PMF of $p_X(0) = q$
$p_X(1) = p$ $\quad p(0)z^0 + p(1)z^1$

$p(0)z^0 + p(1)z^1$
$q + pz$ $\quad$ & PGF is $G_X(z) = q + pz$ $\quad 0 \times p(0) + 1 \times p$

As we know $E[X] = \sum_{x_i=0}^{\infty} x_i\, p_X(x_i)$ $\quad 0 \times p(0) + 1\,p(1) +$
$0^2 p(0) + 1^2 p(1) + 2^2 p$

Mean Value of Bernoulli process = Expectation = $E[X] = 0\cdot p_X(0) + 1\cdot p_X(1)$
$= 0 + p$
$= p$

Now Variance ie $\sigma^2 = E[X^2] - (E[X])^2$
$= p - p^2$

$E[X^2] = \sum x^2 p_X(x)$
$E[X^2] = (0)^2 \times p_X(0) + (1)^2 \times p_X(1)$
$= p$

$$\sigma^2 = p - p^2$$
$$= p(1-p)$$
$$= pq$$

Standard Deviation ie $SD = \sigma = \sqrt{\text{Variance}}$
$= \sqrt{pq}$

Coefficient ie $C_X = \dfrac{\sqrt{\text{Variance}}}{E[X]} = \dfrac{\sqrt{pq}}{p} = \sqrt{\dfrac{q}{p}}$

The Random Variable associated with Bernoulli process in

A) No of successes in n trials has binomial distribution

B) No of trials needed for 1st success has 1st Order Interarrival or Geometric Distribution

c) No of trial needed for r^{th} success has r^{th} order Dist or -ve Binomial Distribution

A) $\boxed{\text{No of successes in } n \text{ trials}}$

Consider a finite or Infinite sequence of Independent R $X_1 X_2 \dots X_n$ then their partial sum is

$$\underline{S_n} = \underbrace{X_1 + X_2 + \dots + X_n} \Rightarrow S_{n-1} = \underline{X_1 + X_2 \dots X_{n-1}} +$$
$$S_n = \underline{S_{n-1}} + X_n$$

And Probability of $S_n(k) =$

$$p[S_n = k / S_{n-1} = k] = p[X_n = 0] = 1-p$$

If There are K successes in n trials

& If we want to find K successes in n trials

trials

sum

$$p[\underline{S_n = k} / \underline{S_{n-1}} = \underline{k-1}] = p[X_n = 1] = p$$

n^{th}

Now Back to Binomial process

PMF of $S_n = k$ = $\boxed{p[S_n = k] = {}^nC_k\, p^k q^{n-k} \quad k = 0, 1, \ldots}$

Also $\displaystyle\sum_{k=0}^{n} {}^nC_k\, p^k q^{n-k} = 1 = (p+q)^n$

$p[S_n = 0] \Rightarrow q^n$

$p[S_n = 1] = np\,q^{n-1}$

$$\boxed{G_X(z) = (q+pz)^n}$$

$$E[X] = \sum_{i=0}^{n} x_i\, f_X(x_i)$$

Remember $\Rightarrow$

$$\left[\; E[X] = \frac{d(G)}{dz}\Big|_{z=1} \qquad E[X^2] = \frac{d^2(G)}{dz^2}\Big|_{z=1} + \frac{dG}{dz}\Big|_{z=1} \;\right]$$

Mean $\Rightarrow$ Now $E[X] = \dfrac{d}{dz}(q+pz)^n\Big|_{z=1}$

$$= n(q+pz)^{n-1}\cdot p\Big|_{z=1}$$

$$= np\,(q+pz)^{n-1}\Big|_{z=1}$$

$$= np\,(q+p)^{n-1}$$

$$= np\,(1)^{n-1}$$

$$= np$$

$np\cdot p(n-1)\,(q+pz)^{n-2}$

$np^2 \cdot n$ $\cancel{(q+p)}$

Variance $\Rightarrow$ $\sigma^2 = E[X^2] - (E[X])^2$

$$E[X^2] = \frac{d^2}{dz^2}(q+pz)^n\Big|_{z=1} + \frac{d}{dz}(q+pz)^n\Big|_{z=1}$$

$$\frac{d}{dz}\left(np\,(q+pz)^{n-1}\right) + np$$

$$= np\,(n-1)p\,(q+pz)^{n-2}\Big|_{z=1} + np$$

$$= n(n-1)p^2 + np$$

$$\therefore \quad \sigma^2 = n(n-1)p^2 + np - n^2p^2$$

$$= np(1-p)$$

$$= npq$$

Standard Deviation $= \sqrt{npq}$

$$\text{Coeff}(X) = \frac{\sqrt{npq}}{np} = \sqrt{\frac{q}{np}}$$

And for every large n; small k, $\lambda t = np$;
Binomial pmf can be approximation by poisson

ie $^{n}C_{k}\, p^k q^{n-k} \approx \dfrac{e^{-a} a^k}{k!}$

B) $\boxed{\text{FIRST ORDER INTER ARRIVAL TIME}}$ (Geometric)

Consider a Bernoulli process $X(n)$ where $n = 1, 2, \ldots n$
If we refer to the success in a Bernoulli process arrival
Then we study the no of trials for the 1st order Inter arrival time

ie If Y_1 denotes the no of trials needed for 1st

$$Y_1 = \left\{ \begin{matrix} Y_0 + Y_1 + Y_2 + \cdots + Y_{k-1} = 0 \\ Y_k = 1 \end{matrix} \right\}$$

Moment: $E[X]$ {1st moment}
$E[X^2]$ {IInd " }

ie called 1st order Interarrival time to be no of trials upto and including 1st success

here Y is also called Geometric PMF with parameter p

$$p_{Y_1}(k) = q^{k-1} p \quad \Big/ \; k = 1, 2, \ldots$$

$$G_{Y_1}(z) = \frac{pz}{1-qz}$$

mean

$$E[Y_1] = \frac{d(G)}{dz}\Big|_{z=1}$$

$$= \frac{d}{dz}\left(\frac{pz}{1-qz}\right)\Big|_{z=1}$$

$$= \frac{(1-qz)p - pz(-q)}{(1-qz)^2}\Big|_{z=1}$$

$$= \frac{p - pqz + pqz}{(1-qz)^2}\Big|_{z=1}$$

$$= \frac{p}{(1-qz)^2}\Big|_{z=1}$$

$$= \frac{p}{(1-q)^2}$$

$$= \frac{p}{p^2}$$

$$= \boxed{\frac{1}{p}}$$

Now $E[Y_1^2] = \frac{d^2(G)}{dz^2} + \frac{d(G)}{dz}$

$$= \frac{d^2}{dz^2}\left(\frac{pz}{1-qz}\right)\Big|_{z=1} + 1/p$$

$$= \sigma \frac{d}{dz}\left(\frac{p}{(1-qz)^2}\right) + 1/p$$

$$= \frac{(1-qz)^2(0) - p\,(2(1-qz)(-q)) }{(1-qz)^4} + 1/p$$

$$= \frac{2pq\,(1-qz)}{(1-qz)^4} + \frac{1}{p}$$

$$= \frac{2p^2q}{p^4} + 1/p$$

$$= \frac{2q}{p^2} + \frac{1}{p}$$

$$\sigma^2 = E[X^2] - (E[X])^2$$

$$= \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{2q-1}{p^2} + \frac{1}{p}$$

$$= \frac{2q-1+p}{p^2}$$

$$= \frac{p+q-1+q}{p^2}$$

$$= \boxed{\frac{q}{p^2}}$$

$$\sigma = \sqrt{\frac{q}{p^2}}$$

$$\text{Coff} = \frac{\sqrt{q}/p^2}{1/p} = \sqrt{q}$$

C) STOCHASTIC PROCESS BASED ON r^{th} ORDER INTER ARRIVAL TIME

Consider Y_r be the no of trial upto and including rth success; then Y_r be termed as r^{th} order interarrival time & PMF of Y_r is

Sol: As we know in -ve binomial

$$pY_r = {}^{n-1}C_{r-1}\, p^r q^{n-r}$$

$$\& \; G_{Y_r}(z) = \left(\frac{pz}{1-qz}\right)^r$$

Mean

Now $E[Y_r] = \dfrac{d}{dz}\left(\dfrac{pz}{1-qz}\right)^r \Big|_{z=1}$

$$= r\left(\frac{pz}{1-qz}\right)^{r-1} \cdot \frac{d}{dz}\left[\frac{pz}{1-qz}\right]_{z=1}$$

$$= r\left(\frac{pz}{1-qz}\right)^{r-1} \cdot \frac{1}{p} \quad \text{at } z=1$$

$$= \frac{r}{p}\left(\frac{p}{1-q}\right)^{r-1}$$

$$= \frac{r}{p}(1)^{r-1}$$

$$= \frac{r}{p}$$

~~Now~~

$$E[Y_r^2] = \frac{d^2}{dz^2}\left(\frac{pz}{1-qz}\right)^r\Big|_{z=1} + \frac{d}{dz}\left(\frac{pz}{1-qz}\right)^r$$

$$= \frac{d}{dz}\left[\frac{r}{p}\left(\frac{pz}{1-qz}\right)^{r-1}\right]_{z=1} + r/p$$

$$= \frac{r}{p}(r-1)\left(\frac{pz}{1-qz}\right)^{r-2}\frac{d}{dz}\left(\frac{pz}{1-qz}\right) + r/p$$

$$= \frac{r(r-1)}{p}(1)^{r-2}\cdot\frac{1}{p} \quad \text{at } z=1$$

$$= \frac{r(r-1)}{p^2} + \frac{r}{p}$$

Variance

$$\sigma^2 = E[Y_r^2] - (E[Y])^2$$

$$= \frac{r(r-1)}{p^2} + \frac{r}{p} - \frac{r^2}{p^2}$$

$$= \frac{r(r-1)}{p^2} + \frac{r}{p} - \frac{r^2}{p^2}$$

$$= \frac{r^2 - r}{p^2} + \frac{r}{p} - \frac{r^2}{p^2}$$

$$= \frac{-r + rp}{p^2}$$

$$= \frac{r(p-1)}{p^2}$$

$$= \frac{-rq}{p^2}$$

Soby

VII) Poisson Process

Poisson process is a continuous parameter discrete state stochastic process ie involved with error or arrival like Bernoulli process.
It's a type of renewal counting process ie in this process we are interested in counting no of events occurring in the time interval 0 to t ie $[0,t]$

For eg: 1) No of incoming telephone calls to a switch board of trunk
2) No of Job arrived to a computer centre

The Poisson process can be constructed as follows

For a value of n with parameter p and k where

$$a = np \quad \text{defined by} \quad {}^{n}C_{k}\, p^{k}\, q^{n-k}$$

Consider if n is very-2 large & k is very-2 small then

$$f_X(x) = \frac{e^{-a}\, a^{x}}{k!} \quad \left(\text{Already proved in unit 2}\right)$$

∴ For the no of events $N(t)$ in the time interval $[0,t]$, has a Poisson Distribution with parameter λt

from the above $f_X(k) = \frac{e^{-a}\, a^{k}}{k!}$ when $a = \lambda t$

& PGF is $e^{-a(1-z)}$

Pg-285

Ques: Show that First Order Inter Arrival time of Non-Homogeneous Bernoulli process is not memoryless

→ 1st order means; here we have to apply Geometric

Some Imp pts

→ In Homogeneous process; our extra time is fixed

→ In Non- " " ; " " " not "

Remember

For Homogeneous Process

⇒ Bernoulli process of 1st order Inter Arrival time; we have $Y=k= q^{k-1}p = (1-p_1)(1-p_2)\cdots(1-p_{k-1})p_k$

For Non-Homogeneous Process

⇒ prob of $p[Y=k+J]$ Condition that / Given that $p[Y>J]$

$$\frac{p[Y=k+J]}{p[Y>J]} = \frac{(1-p_1)(1-p_2)\cdots(1-p_{k+J-1})p_{k+J}}{1-p[Y\le J]}$$

$$= \frac{(1-p_1)(1-p_2)\cdots(1-p_{k+J-1})p_{k+J}}{1-F_Y(J)}$$

$$= \frac{(1-p_1)\cdots\cdots}{1-(1-q^J)} \quad [\because F_X(x) = 1$$

Let $k=1$, $J=1$

then $P[Y=2] = \frac{p[Y=2]}{p[Y>1]}$

$= \frac{q^{2-1}p}{q}$

$= p$

As $p \neq q^{k-1}p$

So Not Memoryless

tean

Now $E[X] = \frac{d}{dz}(G)\Big|_{z=1}$

$$= \frac{d}{dz}\left(e^{-a(1-z)}\right)\Big|_{z=1}$$

$$= \frac{d}{dz}\left(e^{-a+az}\right)$$

$$= \frac{d}{dz}\left(e^{-a} \cdot e^{az}\right)$$

$$= ae^{-a} \cdot e^{az}$$

$$= a \cdot e^{0}$$

$$= \boxed{a}$$

Now $E[X^2] = \frac{d^2}{dz^2} G\Big|_{z=1} + \frac{d}{dz} G\Big|_{z=1}$

$$a^2 + a$$

variance

$$\sigma^2 = E[X^2] - (E[X])^2$$

$$= a^2 + a - a^2$$

$$= a$$

SD

$$SD = \sigma = \sqrt{a}$$

coeff

$$\text{Coeff}(X) = \frac{SD}{EX} = \frac{\sqrt{a}}{a} = \frac{1}{\sqrt{a}}$$

From the above we conclude that

I) Mean & Variance of the Poisson Distⁿ have same,
II) Parameter λ is called arrival rate of poisson process,
III) Poisson process plays an Impt role in queuing theory & Reliability theory

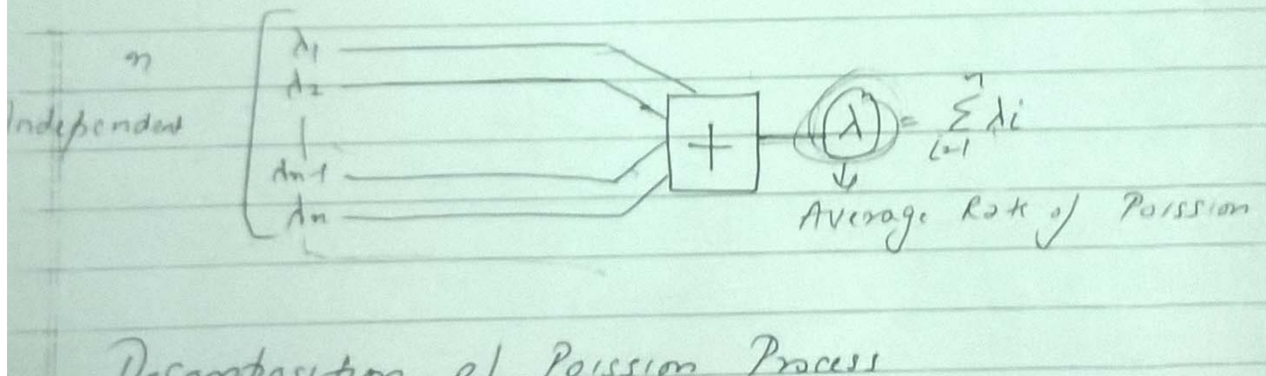
Write a short note on Superposition of Poisson process and Decomposition of Poisson process.

→ ## Superposition of Poisson Process

We know that sum of n Independent Poission R.V is itself a Poisson. Based upon this we can show that superposition of n independent poission process with respective rate $\lambda_1\ \lambda_2\ \cdots\ \lambda_n$ is also a poission process with average rate λ

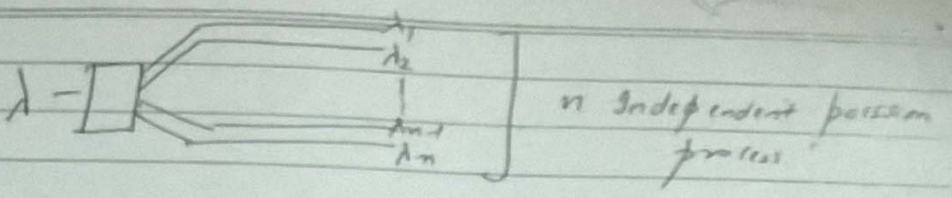
ie $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

The notation of superposition process is shown below



Decomposition of Poission Process

A similar result of Decomposition process ie with average arrival rate λ, branches out with n o/p path as shown in fig below



Renewal Process

Let $X_1 X_2 \ldots X_n$ be n independent Non-Negative R.V with a common Distribution function
Then X_i where $i=1$ to n is called Renewal process

Now consider no of renewal $N(t)$ in the time interval $[0,t]$, Then the discrete state Continuous parameter process $N(t)$ is called Renewal Counting process.

This can be generalised as poisson process related to the R.V S_k and S_{k+1} by observing that $N(t) = n$

Now we observe that Probability of Total no of Renewal

$$\text{ie } p[N(t) = n] \Rightarrow p[S_n < t < S_{n+1}]$$
$$= p[S_n \le t] - p[S_{n+1} \le t]$$
$$= F_n(t) - F_{n+1}(t)$$

$$\text{Also } \left[F_0(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \right] - (3)$$

$F_0(t)$ means Zero Arrival ie
Till Now bulb is glowing

$$\text{As } p(N(t) = n) = F_n(t) - F_{n+1}(t) - (2)$$

$$\text{Illy } p[N(t) = 0] = F_0(t) - F_1(t) - (1).$$

7th <u>Write a short note on Expected No of Renewals</u>

$N(t)$: No of Renewals at time t

$\therefore$ $E[N(t)]$ is called Expected no of renewals and is defined as

$$E[N(t)] = \sum_{n=0}^{\infty} n \cdot p[N(t) = n] \quad \left[\because E[X] = x\,px\right]$$

$$= \sum_{n=0}^{\infty} n\left[F_n(t) - F_{n+1}(t)\right] \quad \text{from eq}^n\ 2$$

$$\therefore E[N(t)] = \sum_{n=0}^{\infty} n F_n(t) - \sum_{n=0}^{\infty} n F_{n+1}(t)$$

$$E[N(t)] = \sum_{n=0}^{\infty} n F_n(t) - \sum_{n=1}^{\infty} (n-1) F_n(t)$$

$$\text{IIly} \quad E[N(t)] = \sum_{n=0} n F_n(t) - \sum_{n=1}^{\infty} (n-1) F_n(t)$$

$$\left[\because \sum_{n=0}^{\infty} n F_{n+1}(t) = \sum_{n=1}^{\infty} (n-1) F_n(t)\right]$$

$$\text{or} \quad E[N(t)] = \sum_{n=1}^{\infty} n F_n(t) - \sum_{n=1}^{\infty} (n-1) F_n(t)$$

$$\because \left[\sum_{n=0}^{\infty} n F_n(t) = \sum_{n=1}^{\infty} n-1\, F_n(t)\right]$$

$$\therefore E[N(t)] = \sum_{n=1}^{\infty} \left[n - (n-1)\right] F_n(t)$$

$$= \sum_{n=1}^{\infty} (1) F_n(t)$$

This is Expected no of Renewal f^n or Renewal $Dist^n$ f^n

$$M[t] = F_1(t) + \sum_{n=1}^{\infty} F_{n+1}(t)$$

Renewal Density Function

It is defined as derivation of Renewal Distribution $M(t)$ and is denoted as $m(t)$

ie $m(t) = \frac{d}{dt} M(t)$

or $m(t) = \frac{d}{dt} \sum_{n=1}^{\infty} F_n(t)$

$$\boxed{m(t) = \frac{d}{dt} \sum_{n=1}^{\infty} F_n(t)}$$

Renewal Model of Program Behaviour

It deals on modelling the memory referencing behaviour of a program executed in paged virtual m/m system.

Consider the logical Address space of N page program ie denoted by $N = [1, 2, 3, \ldots, n]$

Then the behaviour of program is captured by the reference string 'S' = which is given as the sequence $S = \{r_1 r_2 \ldots r_m\}$

where each r_t is in each N, ie

If $r_t = i$, then it means that a reference is made to the page indexed by i at the t^{th} reference.

$S = r_1 r_2 \ldots r_n$

r_t each N

$r_t = i$

1	r_1
2	r_2
3	r_3
4	r_4
	⋮
	r_n

Consider referencing string S is a discrete state, discrete parameter stochastic process, which is decomposed into n distinct stochastic process

Let the 1st order Inter arrival time of i^{th} page, index at reference t is distributed by $F_i(r)$ and $P_i(r)$ is called interference PMF of index i. Then the mean Interference Interval for index i is given by $E_x = \sum_{i=1}^{n} x_i \, p_X(x_i)$

$$E_x = \int_{-\infty}^{\infty} x f_X(x) \, dx$$

Now consider $m_i(r)$ denoted the renewal density of i^{th} process then

$m_i(r)$ is the prob of reference to page i and is given as $m_i(r) = \frac{1}{E[X]}$

The Virtual m/m system usually retains only a portion of program logical address space in the main m/m.

For Each Instant of time; address space in the main m/m is determined by paging algorithm

The main paging algorithm is the Working set Alg.

Working set Algorithm — ie how pages are referenced in m/m?

→ A working set $W(t, \tau)$ at time t is defined as the set of distinct pages referenced in the time interval $(0, t)$.
Here τ is known as window size.

→ Working set algorithm assures that a page in the working set at time is in the main m/m

→ If the next page ie to be referenced is not in the working set ie $r(t+1) \notin W(t, \tau)$
Then it means that page fault has occured and required page will be resend to O.S again

* Measurement of W.S Algorithm

The important measurement of W.S algo are the asymptotic average W.S size (s) of τ ie $S(\tau)$

In order to compute $S(\tau)$; consider a random incidence in an inter reference interval of page i
then $S(\tau) = \lim_{t \to \infty} E[W(t, \tau)]$

Now consider T_i be the backward recurrence time, then by using Backward Analog Eq^n

Av. size is $S(\tau) = \sum_{i=1}^{n} \sum_{j=0}^{\tau-1} \frac{1 - F_i(j)}{E[X_i]}$

#. Average Page Fault Rate

The conditional probability of page fault given that the i^{th} page is referenced at time t is $1 - F_i(\tau)$.

So this is the probability when the page fault i.e. the next page to be referenced exceeds window size.

Now By using the WS Algorithm, we can calculate the Average Page fault Rate

$$\phi(\tau) \text{ i.e. } \phi(\tau) = \sum_{i=1}^{n} \frac{1 - F_i(\tau)}{E[X_i]}$$

Que 1) In this ques; Do proof of Markov which we have done in geometric.

If same ques comes for continuous thie do proof of markov in exponential.

Que 2) (i) As we Know $p_X(x) = \frac{e^{-a} a^k}{k!}$

Consider the no of arrival at time t is $n(t)$ is poission distributed with parameter λt

Now compute the probability that No of arrival at time t is even ie $p[N_t = \text{even}]$

$$\Rightarrow \sum_{i=0}^{n} p[N_t = 2i]$$

$$= \sum_{i=0}^{n} \frac{e^{-\lambda t}(\lambda t)^{2i}}{2i!}$$

$$= \frac{1}{2} \sum_{i=0}^{n} \frac{2e^{-\lambda t}(\lambda t)^{2i}}{2i!}$$

$$= \frac{e^{-\lambda t}}{2}\left[\sum_{i=0}^{n} \frac{(\lambda t)^i}{i!} + \sum_{i=0}^{n} \frac{(-1)^i (\lambda t)^i}{i!}\right]$$

$$= \frac{e^{-\lambda t}}{2}\left[e^{\lambda t} + e^{-\lambda t}\right]$$

$$\left[\because \frac{(\lambda t)^0}{0!} - \frac{(\lambda t)^1}{1!} + \frac{(\lambda t)^2}{2!} - \cdots = e^{-\lambda t}\right]$$

$$\& \left[\frac{(\lambda t)^0}{0!} + \frac{(\lambda t)^1}{1!} + \frac{(\lambda t)^2}{2!} + \cdots = e^{\lambda t}\right]$$

$$\Rightarrow \frac{e^{t} + e^{-2\lambda t}}{2}$$

$$= \boxed{\frac{1 + e^{-2\lambda t}}{2}}$$

(ii) Compute the prob when no of arrivals at t is odd ie $p[N_t = \text{odd}]$

$$\Rightarrow \sum_{i=0}^{n} p[N_t = (2i)+1]$$

$$\Rightarrow \sum_{i=0}^{n} \frac{e^{-\lambda t}(\lambda t)^{2i+1}}{(2i+1)!}$$

$$= \frac{1}{2} \sum_{i=0}^{n} \frac{2e^{-\lambda t}(\lambda t)^{2i+1}}{(2i+1)!}$$

$$= \frac{1}{2} e^{-\lambda t} \left[\sum_{i=0}^{n} \frac{(\lambda t)^i}{i!} - \sum_{i=0}^{n} (-1)^i \frac{(\lambda t)^i}{i!} \right]$$

$$= \frac{1}{2} e^{-\lambda t} \left[e^{\lambda t} - e^{-\lambda t} \right]$$

$$= \frac{1 - e^{-2\lambda t}}{2}$$

ue 3 As we know $E[X] = \sum_{i=1}^{n} x_i \, p_X(x_i)$

$$= 3 \times \frac{1}{3} + 3\cos(t) \times \frac{1}{3} + 4\sin(t$$

$$\frac{(\lambda t)^0}{0!} + \cancel{\frac{(\lambda t)^1}{1!}} + \frac{(\lambda t)^2}{2!} - \quad = 1 + \cos t\,(t) + \frac{4}{3}\sin(t)$$

$$+ \frac{(\lambda t)^0}{0!} - \cancel{\frac{(\lambda t)^1}{1!}} + \frac{(\lambda t)^2}{2!}$$

$$2\left[\frac{\lambda t^0}{0!} + \frac{\lambda t^2}{2!} + \ldots\right.$$

$R(t_1 t_2) = E[X(t_1) X(t_2)]$

$= (3)(3)(1/3) + (3\cos(t_1))(3\cos(t_2)) \times 1/3 + 4\sin(t_1)\, 4\sin(t_2)\,(1/3)$

$= 9/3 + 9/3 \cos(t_1)\cos(t_2) + 16/3 \sin(t_1)\sin(t_2)$

As In $E[X]$ If we add some time in t
then value of $E[X]$ changes
So it is not strictly stationary

As in $R(t_1 t_2)$ we have 2 time
So we add some time in t_1 t_2
Then value of $R(t_1 t_2)$ changes
So it is not Wide Sense stationary

Ans 4) → Doubt

Do theorm 6.1 by yourself ⇒ VV Imp ⇒ do come Xm