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SMCS

UNIT-2

DISCRETE RANDOM VARIABLES

RANDOM VARIABLE:

	X(success)	Y(failure)	Z(sum)
000	0	3	3
001	1	2	3
010	1	2	3
011	2	1	3
100	1	2	3
101	2	1	3
110	2	1	3
111	3	0	3

Definition: A Random variable X on a sample space S is a function $X : S \to \mathbb{R}$ that assigns a real no. $X(s)$ to each sample point $s \in S$.

In the above example, $X \in \{0, 1, 2, 3\}$

EVENT SPACE:

$A_0 = \{000\}$

$A_1 = \{(001), (010), (100)\}$

$A_2 = \{(011), (101), (110)\}$

$A_3 = \{111\}$

For a Random Variable X and a real no. x, we define the events A_x as

$$A_x = \{ s \in S / X(s) = x \}$$

value, S.point, Sspace, Random Variable

These events are mutually exclusive as well as collectively exhaustive events.
Thus collection of events A_x for all x defines an EVENT SPACE.

Probability Mass function (pmf):
(exactly)

$P(A_0) = 1/8$ $\quad P_X(0) = P(X=0)$
$P(A_1) = 3/8$ $\quad P_X(1) = P(X=1)$
$P(A_2) = 3/8$ $\quad P_X(2) = P(X=2)$
$P(A_3) = 1/8$ $\quad P_X(3) = P(X=3)$

$\Rightarrow \boxed{P_X(x) = P(X=x)}$

Properties of pmf :-

1. Since $P_X(x)$ is a probability
   $\therefore 0 \leq P_X(x) \leq 1$

2. Since the random variable assigns some value $x \in \mathbb{R}$ to each sample point $s \in S$, we must have
   $$\sum_{x \in \mathbb{R}} P_X(x) = 1$$

3. The property above may be restated as:
   $$\sum_{P_X(x)} P_X(x) = 1$$

Probability distribution function (Pdf):
(atmost)

- $F_X(2) = P_X(0) + P_X(1) + P_X(2)$
  $= F_X \; P(X \leq 2)$
  $\boxed{F_X(t) = P(X \leq t)}$

- $P(X > 3) = 1 - F_X(3)$
  $\boxed{P(X > t) = 1 - F_X(t)}$

- $\boxed{P(X \geq t) = 1 - F_X(t-1)}$

- $P(2 < X \leq 5) = F_X(5) - F_X(2)$

$\boxed{P(a < X \leq b) = F_X(b) - F_X(1)}$

- $P(2 \leq X \leq 5) = F_X(5) - F_X(1)$

$\boxed{P(a \leq X \leq b) = F_X(b) - F_X(a-1)}$

Properties:

1. Since $F_X(x)$ is a probability
$\therefore 0 \le F_X(x) \le 1.$

2. $\lim_{x \to -\infty} F_X(x) = 0$

$\lim_{x \to \infty} F_X(x) = 1$

3. $F_X(x)$ is a monotonic non-decreasing function.
$x_2 > x_1 \qquad F_X(x_2) \geq F_X(x_1)$

4. Positive jump
$F_X(5) = F_X(4) + P_X(5)$

$$\boxed{F_X(x_i+1) = F_X(x_i) + P_X(x_i+1)}$$

Bernoulli pmf:

$P_X(0) = P(X=0) = q$ (prob. of failure)
$P_X(1) = P(X=1) = p$ (prob. of success)

Bernoulli pdf:

$F_X(0) = P_X(0) = q$
$F_X(1) = P_X(0) + P_X(1)$
$= q + p$
$= 1$

Binomial pmf:

$$\boxed{P(X=k) = P_X(k) = \binom{n}{k} p^k . q^{n-k}}$$

This is also called as binomial density $b(k,n,p)$
i.e. prob. of k successes in n trials with p as ~~the~~ prob. of success.
Here, n and p are parameters of binomial density

Binomial pdf:

$$\boxed{F_X(t) = \sum_{k=0}^{t} \binom{n}{k} p^k . q^{n-k}}$$

This is called binomial distribution $B(t,n,p)$.
Here, n and p are parameters of binomial distribution

example 2.4 Consider a plant manufacturing IC chips

10% defective i.e. $p = 0.10$

no. of trials $(n) = 35$

find k (no. of successes)

$$b(k; n, p) = \binom{n}{k} p^k \cdot q^{n-k}$$

$$b(k; 35, 0.1) = \binom{35}{k} 0.1^k (0.9)^{35-k}$$

example 2.5 n-components

binomial distribution $b(m; n, R)$

R is the reliability of a single component

Reliability of m out of n system is given by

$$R_{m/n} = \sum_{k=m}^{n} \binom{n}{k} R^k (1-R)^{n-k}$$

$$= 1 - \sum_{k=0}^{m-1} \binom{n}{k} R^k (1-R)^{n-k}$$

$$= 1 - F(m-1)$$

example 2.6 n-transmitted digits

no. of digits received correctly = C_n

$P(C) = p$ (p is the prob. of successfully transmitting 1 digit)

$P(\bar{C}) = 1-p$

Prob. of exactly i errors is given by

$$P_e(i) = \binom{n}{i} (1-p)^i \; p^{n-i}$$

Prob. of error-free transmission is given by

$$P_e(0) = \binom{n}{0} (1-p)^0 \; p^{n-0}$$

$$= p^n$$

example 2.7 10-IC chips

If no chip is defective, the entire batch is accepted, otherwise rejected.

pmf of defective chips is $b(k; 10, p)$ where p is the prob. that a randomly chosen chip is defective.

Find the prob that the entire batch is accepted.

Sol: A batch is accepted if there is no defective chip i.e. $k=0$

$$\therefore \; b(0; 10, p) = \binom{10}{0} p^0 (1-p)^{10-0}$$

$$= (1-p)^{10}$$

Geometric pmf :

We consider a sequence of Bernoulli trials, but instead of counting no. of successes in a fixed number n of trials, we count the no. of trials until the first success occurs.
The sample space has countably infinite no. of sample points.

Let Z be a Random variable denoting success at i^{th} level.

$Z \in \{1, 2, 3, \ldots\}$

To find the pmf of Z, we note that the event $\{Z = i\}$ occurs iff we have a sequence of $i-1$ failures followed by one success

$$\boxed{P_Z(i) = p q^{i-1}}$$

By the formula for the sum of geometric series, we have

$$\boxed{\frac{a}{1-r}}$$

$$\sum_{i=1}^{\infty} P_Z(i) = \sum_{i=1}^{\infty} p \cdot q^{i-1}$$

$$= p \sum_{i=1}^{\infty} q^{i-1}$$

$$= \cancel{p}\left[\frac{1}{\cancel{1-q}}\right] = 1$$

→ Distribution function of Z:

$$F_Z(t) = \sum_{i=1}^{t} p \cdot q^{i-1} = p \sum_{i=1}^{t} q^{i-1}$$

$$= \cancel{p}\left(\frac{1-q^t}{\cancel{1-q}}\right)$$

$$\boxed{\frac{a(1-r^n)}{1-r}}$$

$$\boxed{F_Z(t) = 1 - q^t}$$

→ Modified Geometric pmf :

The Random variable Z counts the total no. of trials upto and including first success.
Now, we are interested in finding no. of failures before first success.

Let X be the Random variable denoting no. of failures until first success.

$X \in \{0, 1, 2, \ldots\}$

So modified geometric pmf is specified by:

$$P_X(i) = p \cdot q^{i-1} \cdot q$$

$$\boxed{P_X(i) = p \cdot q^i}$$

⇒ ~~dist.~~

→ Distribution function of X:

$$F_X(t) = \sum_{i=0}^{t} p \cdot q^i$$

$$= p \sum_{i=0}^{t} q^i$$

$$= \frac{p(1-q^{t+1})}{\cancel{1-q}}$$

$$\boxed{F_X(t) = 1-q^{t+1}}$$

Markov property of Geometric pmf:

Consider a sequence of n bernoulli trials
Let 'Z' represent the no. of trials until first success
Assume we have observed a fixed no. 'n' of these trials and found them all to be failures.
Let 'Y' denote the no. of additional trials to reach the success.
Then,

$$Z = Y + n \qquad Z > n$$

Conditional probability is

$$q_i = P(Y=i \mid Z>n)$$

$$= P(Z-n=i \mid Z>n)$$

$$= P(Z=n+i \mid Z>n)$$

$$= \frac{P(Z=n+i \;\&\; Z>n)}{P(Z>n)}$$

$$= \frac{P(Z=n+i)}{P(Z>n)}$$

$$= \frac{P_Z(n+i)}{1-F_Z(n)}$$

$$= \frac{p \cdot q^{n+i-1}}{1-1+q^n}$$

$$= p \cdot q^{i-1}$$

Negative Binomial pmf :

let T_r be the Random variable denoting no. of trials required to get 'r' successes.

Then,

$T_r \in \{r, r+1, r+2, \ldots\}$

Let To compute $pT_r(n)$, define the following events

A = "$T_r = n$" ; n trials are required to get r successes

B = n-1 trials result in r-1 success

C = n^{th} trial should result in a success

Then, $P(A) = P(B \cap C)$
$= P(B) \cdot P(C)$

$$P(B) = \binom{n-1}{r-1} p^{r-1} \cdot q^{n-(r+1)}$$

$$\boxed{P(B) = \binom{n-1}{r-1} p^{r-1} \cdot q^{n-r}}$$

$$\boxed{P(C) = p}$$

Now,

$$pT_r(n) = P(A) = P(B) \cdot P(C)$$
$$= \binom{n-1}{r-1} p^{r-1} \cdot q^{n-r} \cdot p$$

$$\boxed{pT_r(n) = \binom{n-1}{r-1} p^r \cdot q^{n-r}}$$

An alternative form of this pmf can be established

$$\boxed{pT_r(n) = P(T_r = n) = \binom{-r}{n-r} (-1)^{n-r} \cdot p^r \cdot q^{n-r}}$$

$n=5$
$r=3$ $\binom{4}{2} = 6$

$\binom{-3}{2} (-1)^2 = \frac{-3!}{2! \, -5!} = \frac{-3 \times -4 \times -5!}{2! \, -5!}$
$= 6$

Poisson pmf:

Consider the no. of job arrivals to a computer center in the time of interval $[0,t]$

Suppose the interval $[0,t]$ is divided into n subintervals of length $\frac{t}{n}$.

Let λ be the job arrival rate.

prob of success $(p) = \frac{\lambda t}{n}$

prob of failure $(q) = \left(1-\frac{\lambda t}{n}\right)$

So the Binomial pmf can be given as:

$$p(k) = \binom{n}{k}\left(\frac{\lambda t}{n}\right)^k \left[1-\frac{\lambda t}{n}\right]^{n-k}$$

$$= \frac{n!}{k!(n-k)!} \cdot \frac{(\lambda t)^k}{n^k} \cdot \left(1-\frac{\lambda t}{n}\right)^n \cdot \left(1-\frac{\lambda t}{n}\right)^{-k}$$

$$= \frac{n(n-1)(n-2)\ldots(n-k+1)\cancel{(n-k)!}}{k!\cancel{(n-k)!}} \frac{(\lambda t)^k}{n^k} \cdot \left(1-\frac{\lambda t}{n}\right)^n \left(1-\frac{\lambda t}{n}\right)^{-k}$$

$$= \frac{n(n-1)\ldots(n-k+1)}{n^k} \cdot \frac{(\lambda t)^k}{k!} \cdot \left(1-\frac{\lambda t}{n}\right)^n \left(1-\frac{\lambda t}{n}\right)^{-k}$$

$$= \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right) \ldots \left(\frac{n-k+1}{n}\right) \cdot \frac{(\lambda t)^k}{k!} \left(1-\frac{\lambda t}{n}\right)^n \left(1-\frac{\lambda t}{n}\right)^{-k}$$

$$= 1\left(1-\frac{1}{n}\right) \ldots \left(\frac{n-k+1}{n}\right) \cdot \frac{(\lambda t)^k}{k!} \left(1-\frac{\lambda t}{n}\right)^n \left(1-\frac{\lambda t}{n}\right)^{-k}$$

Now, for a subinterval width to be very small (i.e. approaches to zero), the value of n should be very large (i.e. $n \to \infty$).

$\therefore$ using $\lim_{n\to\infty}$ we get

$$\lim_{n\to\infty} \frac{(\lambda t)^k}{k!} \left(1-\frac{\lambda t}{n}\right)^n$$

$$\boxed{\text{Let } \frac{-\lambda t}{n} = h \quad \text{where } n \to \infty,\ h \to 0 \qquad n = \frac{-\lambda t}{h}}$$

$$\frac{(\lambda t)^k}{k!} \lim_{h\to 0} \left[[1+h]^{1/h}\right]^{-\lambda t}$$

So, $\boxed{f(k;\lambda t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}}$

Put $\alpha = \lambda t$

$$\boxed{f(k;\alpha) = \frac{\alpha^k}{k!} e^{-\alpha}}$$

Thus, the poisson pmf can be used as a

convenient approximation to the binomial pmf when n is large and p is small.

ie when $n \geq 20$ and $p \leq 0.05$

otherwise binomial.

example 2.9 100-chips

P(defective) = 0.01

P(non-defective) = 0.99

Find the probability that from the box no defectives are found.

sol^n with binomial

$$b(0;100,0.01) = \binom{100}{0}(0.01)^0(0.99)^{100}$$

but for $n \geq 20$ we apply poisson and here $n=100$

$$\alpha = \lambda t = n \times p = 100 \times 0.01 = 1$$

$$P(\text{no defectives are found}) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$f(0;1) = \frac{e^{-1}(1)^0}{0!}$$

Hypergeometric pmf:

(pmf without replacement)

In hypergeometric pmf $h(k;n,d,N)$, we are interested in the probability of choosing k defective components in a random sample of n components, chosen without replacement, from a total of N components, d of which are defective.

$\therefore$ No. of sample points $= \binom{N}{n}$

k defectives can be selected from d defectives in $\binom{d}{k}$ ways.

$(n-k)$ non-defectives can be selected from $(N-d)$ non-defectives in $\binom{N-d}{n-k}$ ways.

$\therefore$ Total no. of favourable cases $= \binom{d}{k}\binom{N-d}{n-k}$

$$\Rightarrow \boxed{h(k;n,d,N) = \frac{\binom{d}{k}\binom{N-d}{n-k}}{\binom{N}{n}}}$$

Example 2.10: Compute the probability of obtaining 3 defectives in a sample of size 10 taken without replacement from a box of 20 components containing 4 defectives.

Solⁿ

16 — 20 — 4
|
7 — 10 — 3

$$h(3; 10, 4, 20) = \frac{\binom{4}{3}\binom{16}{7}}{\binom{20}{10}}$$

$$= 0.247678$$

Problems:

1. 10-jobs
6- class I   4 - class(II)
A Random sample of size n is selected
let X be the no. of class I jobs in the sample
Calculate pmf of X if the sampling is
a) without replacement

P(X=k) = 4 — 10 — 6
|
n-k — n — k

$$P(X=k) = \frac{\binom{4}{n-k}\binom{6}{k}}{\binom{10}{n}}$$

b) with replacement

$$P(X=k) = \binom{n}{k}(0.6)^{k}(0.4)^{n-k}$$

2. n- equally likely passwords
N_n = no. of trials
Determine pmf of N_n.
a) If unsuccessful passwords are not eliminated from further selection.
(with replacement)

P(N

b) If unsuccessful passwords are eliminated from further selections. (without replacement)

Sol^n $p = \frac{1}{n}$ $q = \left(1 - \frac{1}{n}\right)$

a) with replacement.

since we have to calculate pmf of Nn ie no. of trials required to break the file.

So, we will apply Geometric pmf

$p(Z_i) = p \cdot q^{i-1}$

$\therefore P(N=k) = \left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^{k-1}$

Because if total no. of trials required to break the password are K then (k-1) are unsuccessful passwords (failures) and the last one is a successful password (success).

b) without replacement

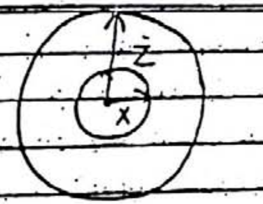
$P(N=k) = \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n-1}\right)\left(1-\frac{1}{n-2}\right) \cdots$

$\left(1-\frac{1}{n-k+1}\right)\left(\frac{1}{n-k}\right)$

$= \left(\frac{\cancel{n-1}}{n}\right)\left(\frac{\cancel{n-2}}{\cancel{n-1}}\right)\left(\frac{n-3}{\cancel{n-2}}\right) \cdots \left(\frac{\cancel{n-k+1-1}}{n-k+1}\right)\left(\frac{1}{\cancel{n-k}}\right)$

$= \frac{1}{n}$

3.



X is a Modified Geometric Random Variable with parameter p.

Z has simple geometric pmf and is the no. of trunks used for a call directed to a destination outside the caller's local exchange.

Determine pmf of Z, given that a call requires atleast 3 trunks.

Sol^n $P(Z=k \mid Z \geq 3) = \frac{P(Z=k \ \& \ Z \geq 3)}{P(Z \geq 3)}$

$= \frac{P(Z=k)}{P(Z \geq 3)}$

$= \frac{P_Z(k)}{1-F_Z(2)}$

$= \frac{p \cdot q^{k-1}}{1-(1-q^2)}$

$= \frac{p \cdot q^{k-1}}{\cancel{1}-\cancel{1}+q^2}$

$= p \cdot q^{k-3}$

4. Prob. of error-free transmission of a message over a comm. channel is p.
If message is not received correctly, a re-transmission is initiated.

$P(\text{error}) = p$

$P(\text{no error}) = 1-p$

a) Find the prob. that no retransmissions are required.

$P(\text{No retransmission}) = p(1-p)^0$
$= p$

b) Find the prob. that exactly 2 retransmissions are required.

$P(2\ \text{retransmissions}) = p\,(1-p)^2$

5. Prob. that jobs need to wait until weekend for scheduling $= 0.01$
Prob. that jobs need not to wait until weekends for scheduling $= 1-0.01$
$= 0.99$

Find the prob. that among a sample of 200 jobs, there are no jobs that have to wait until the weekend for scheduling.

Sol^n Acc. to Binomial distribution

$P(\text{no job have to wait}) = \binom{200}{0}(0.01)^0(0.99)^{200}$

But, for $n \geq 20$ we use POISSON and here $n = 200$.

∴ Acc. to Poisson distribution

$\alpha = \lambda t = np$
$= 200 \times 0.01 = 2$

$P(\text{no job have to wait}) = e^{-\alpha}\dfrac{\alpha^k}{k!}$

$= e^{-2}\dfrac{2^0}{0!} = 0.1353$

6. 5% of the disk controllers produced by a plant are known to be defective i.e. $P(D) = 0.05$
$P(\bar{D}) = 0.95$

A sample of 15 controllers is drawn randomly from each month's production and no. of defectives noted.
Find the prob. that these monthly samples would have atleast 2 defective controllers.

Sol^n $P(D) = 0.05$ $P(\bar{D}) = 0.95$

$P(\text{atleast 2 defective controllers}) = \sum_{k=2}^{15}\binom{15}{k}(0.05)^k(0.95)^{15-k}$

$= 1 - \sum_{k=0}^{1}\binom{15}{k}(0.05)^k\,(0.95)^{15-k}$

$= 1-(0.95)^{15} - 15 \times 0.05 \times (0.95)^{14}$

$= 0.1709525$

7. Prob. of error in the transmission of a bit is; $p = 10^{-4}$
$= 0.0001$
Find the prob. of more than 3 errors in transmitting a block of 1000 bits.

Soln $P(E) = 0.0001$ $P(\bar{E}) = 0.9999$
Acc. to poisson distribution
$\alpha = \lambda t = np$
$= 1000 \times 0.0001$
$= 0.1$
$P(\text{more than 3 errors}) = 1 - \sum_{k=0}^{3} \frac{e^{0.1} (0.1)^k}{k!}$
$= 3.825 \times 10^{-6}$

8. Interval = t seconds
parameter = 0.3 t
Find the prob. of the following events
a) exactly 3 messages will arrive during 10 sec interval.
b) atmost 20 messages will arrive in 20 sec.
c) No. of message arrivals in an interval of duration 5 sec is between 3 and 7.

Soln Acc. to Poisson distribution,
$\alpha = \lambda t$
$= 0.3t$
$\Rightarrow \lambda = 0.3$

a) $k=3$ $t = 10$
$\lambda t = 0.3 \times 10 = 3$
P(exactly 3 messages arrive in 10 sec interval)
$= \frac{e^{-3} (3)^3}{3!}$
$= 0.224$

b) $0 \leq k \leq 20$ $t = 20$
$\lambda t = 0.3 \times 20 = 6$
P(atmost 20 messages arrive in 20 sec)
$= \sum_{k=0}^{20} \frac{e^{-6} (6)^k}{k!}$
$= \cancel{0.99}$ $= 0.973.$

c) $3 \leq k \leq 7$ $t = 5$
$\lambda t = 0.3 \times 5 = 1.5$
P(no. of message arrivals in an interval of duration 5 sec is between 3 and 7)
$= \sum_{k=3}^{7} \frac{e^{-1.5} (1.5)^k}{k!}$
$= 0.1909836$

9. ISI chip fail with the rate of 1 chip in about 5 weeks. If there are 2 spare chips on hand, and if a new supply will arrive in 8 weeks, find the prob. that during the next 8 weeks, the system will be down for a week or more, owing to lack of chips.

Sol^n Failure rate λ = 1 chip in 5 weeks

$$= \frac{1}{5}$$

$$= 0.2 \text{ chips/week}$$

$$\alpha = \lambda t$$
$$= 0.2 \times 7$$
$$= 1.4$$

P (system is down for atleast 1 week before new supply in 8 weeks) = P (3 or more failures in 7 weeks)

$$= 1 - \sum_{k=0}^{2} \frac{e^{-1.4}(1.4)^k}{k!}$$

$$= 0.1665$$

Probability Generating Function:

The Probability Generating function of a discrete random variable X is defined as:

$$G_X(z) = P_X(0)z^0 + P_X(1)z^1 + P_X(2)z^2 + \cdots$$

$$= \sum_{i=0}^{\infty} P_X(i)z^i, \quad \text{where } P_X(i) \text{ represents pmf.}$$

$G_X(z)$ is also known as z-transform of X which is a convenient tool that simplifies the computations involving integer-valued, discrete random variables.

Theorem: If two discrete random variables X and Y have the same PGf's, then they must have the same distributions and pmf's.

Proof: $G_X(z) = G_Y(z)$ (given)

Now,

$$G_X(z) = P_X(0)z^0 + P_X(1)z^1 + P_X(2)z^2 + \cdots \quad -(1)$$
$$G_Y(z) = P_Y(0)z^0 + P_Y(1)z^1 + P_Y(2)z^2 + \cdots \quad -(2)$$

comparing coefficients of z in (1) and (2)

$$P_X(0) = P_Y(0)$$
$$P_X(1) = P_Y(1)$$
$$\vdots \qquad \vdots$$

$\Rightarrow$ X and Y have same pmf's.

Also, $F_X(t) = P_X(0) + P_X(1) + P_X(2) + \dots + P_X(t)$
$= P_Y(0) + P_Y(1) + P_Y(2) + \dots + P_Y(t)$
$= F_Y(t)$

Thus, X and Y have the same pdf's also.

Hence Proved.

→ **PGF's of Some Discrete Distributions :-**

1. **The Bernoulli Random Variable:**

$$G(z) = P_X(0)z^0 + P_X(1)z^1$$
$$= q + pz$$
$$= 1 - p + pz$$

2. **The Binomial Random Variable:**

$$G(z) = \sum_{k=0}^{n} \binom{n}{k} p^k \cdot q^{n-k} \cdot z^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} (pz)^k \cdot q^{n-k}$$

$$= (pz+q)^n$$

3. **The Modified Geometric Random Variable:**

$$G(z) = \sum_{i=1}^{\infty} p \cdot q^i z^i$$

$$= p \sum_{i=1}^{\infty} (qz)^i$$

$$= \frac{p}{1-qz}$$

4. **The Poisson Random Variable:**

$$G(z) = \sum_{k=0}^{\infty} \frac{\alpha^k \cdot e^{-\alpha}}{k!} \cdot z^k$$

$$= e^{-\alpha} \sum_{k=0}^{\infty} \frac{(\alpha z)^k}{k!}$$

$$= e^{-\alpha} e^{\alpha z}$$
$$= e^{-\alpha(1-z)}$$

(unsolved) ques?

5. **Simple Geometric Random Variable:**

$$G(z) = \sum_{i=1}^{\infty} p \cdot q^{i-1} z^i$$

$$= pz \sum_{i=1}^{\infty} (qz)^{i-1}$$

$$= \frac{pz}{1-qz}$$

6 (que 3) **The Negative Binomial Random Variable:**

$$G(z) = \sum_{n=r}^{\infty} \binom{-r}{n-r} (-1)^{n-r} \cdot p^r \cdot q^{n-r} \cdot z^n$$

$$= p^r \cdot z^r \sum_{n-r=0}^{\infty} \binom{-r}{n-r} (-1)^{n-r} \cdot q^{n-r} \cdot z^{n-r}$$

put $n-r=j$

$$= (pz)^r \sum_{j=0}^{\infty} \binom{-r}{j} (-1)^j (qz)^j$$

$$= (pz)^r \sum_{j=0}^{\infty} \binom{-r}{j} (-qz)^j$$

$$= (pz)^r (1-qz)^{-r}$$

$$= \left(\frac{pz}{1-qz}\right)^r$$

Discrete Random Vector:-

Let $X_1, X_2, \ldots, X_r$ be r discrete Random variables defined on a sample space S. Then for each $s \in S$,
$X_1(s) = x_1$, $X_2(s) = x_2, \ldots$ $X_r(s) = x_r$
and,
$X = (X_1, X_2, \ldots, X_r)$ is a r-dimensional discrete random vector valued function
$X: S \to \mathbb{R}^r$ with $X(s) = (x_1, x_2, \ldots, x_r)$.

→ **Joint pmf:**

Joint or compound pmf is defined to be:
$P(X=x) = P(X_1(s)=x_1, X_2(s)=x_2, \ldots X_r(s)=x_r)$

Properties:
same as pmf.

example:

x \ y	1	2	3	4
1	1/4	1/16	1/16	1/8
2	1/16	1/8	1/4	1/16

$P_{x,y}(1,1) = 1/4$ $\quad$ $P_{x,y}(1,2) = 1/16$

→ Marginal pmf:

$P_X(1) = P(1,1) + P(1,2) + P(1,3) + P(1,4)$

$= \frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$

$= \frac{1}{2}$

$P_Y(1) = P(1,1) + P(2,1)$

$= \frac{1}{4} + \frac{1}{16}$

$= \frac{5}{16}$

$P_X(2) = P(2,1) + P(2,2) + P(2,3) + P(2,4)$

$= \frac{1}{2}$

$P_Y(2) = P(1,2) + P(2,2)$

$= \frac{3}{16}$

$P_Y(3) = P(1,3) + P(2,3)$

$= \frac{5}{16}$

$P_Y(4) = P(1,4) + P(2,4)$

$= \frac{3}{16}$

example:

x \ y	1	2
1	a	$\frac{1}{2} - a$
2	$\frac{1}{2} - a$	a

$P_X(1) = P_X(1,1) + P_X(1,2)$

$= a + \frac{1}{2} - a$

$= \frac{1}{2}$

$P_X(2) = P_X(2,1) + P_X(2,2)$

$= \frac{1}{2} - a + a$

$= \frac{1}{2}$

$P_Y(1) = \frac{1}{2}$

$P_Y(2) = \frac{1}{2}$

Generalized Bernoulli Trial:

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}\,p_1^{n_1}\,p_2^{n_2}\cdots p_k^{n_k}$$

example 2.12 5 chips

P(accepting the chip) = 0.70

P_2(rejecting the chip) = 0.20

P_3(submitting chip for reinspection) = 0.10

find the prob. that

~~a) P(all chips must be reinspected)~~

$$P = \frac{5!}{n_1!\,n_2!\,n_3!}(0.70)^{n_1}(0.20)^{n_2}(0.10)^{n_3}$$

$$n_1+n_2+n_3=n$$

$$p_1+p_2+p_3=p$$

a) P(all chips must be reinspected)

$$= \frac{5!}{0!\,0!\,5!}(0.70)^0(0.20)^0(0.10)^5$$

$n_1=0$

$n_2=0$

$n_3=5$

$$= 10^{-5}$$

b) P(none of the chips must be reinspected)

$n_1+n_2=5$

$n_3=0$

$$\sum_{n_1+n_2=5}\frac{5!}{n_1!\,n_2!}(0.70)^{n_1}(0.20)^{n_2}$$

$$= \sum_{n_1+n_2=5}\frac{5!}{n_1!\,(5-n_1)!}(0.70)^{n_1}(0.20)^{5-n_1}$$

$$= \sum_{n_1+n_2=5}\binom{5}{n_1}(0.70)^{n_1}(0.20)^{5-n_1}$$

$$= (0.70+0.20)^5$$

$$= 0.59$$

c) P(atleast one of the chip must be reinspected)

= 1 − P(no chip is reinspected)

= 1 − 0.59

= 0.41

Problems:

①

x \ y	1	2	3
1	1/12	1/6	1/12
2	1/6	1/4	1/12
3	1/12	1/12	0

a) $P(X \le 1\tfrac{1}{2}) = P(X=1)$

$= P(1,1) + P(1,2) + P(1,3)$

$= \frac{1}{3}$

P(X is odd)

b) $P(X \text{ is odd}) = P(X=1) + P(X=3)$
$= P(1,1) + P(1,2) + P(1,3) + P(3,1) + P(3,2) + P(3,3)$
$= \frac{1}{2}$

c) $P(XY \text{ is even}) = P(1,2) + P(2,1) + P(2,2) + P(2,3) + P(3,2)$
$= \frac{3}{4}$

d) $P(Y \text{ is odd} / X \text{ is odd}) = \frac{P(Y \text{ is odd} \;\&\; X \text{ is odd})}{P(X \text{ is odd})}$

$= \frac{P(1,1) + P(1,3) + P(3,1) + P(3,3)}{(b)}$

$= \frac{1}{2}$ Ans

example 2.8: Simpson's Reversal Paradox

Consider 2 shipments (labeled I and II) of 'r' chips from each of the 2 manufacturers A and B.
According to Simpson's Reversal paradox, if we consider the 2 shipments individually and manufacturer B is sending better chips than manufacturer A then mixing the 2 shipments & subsequent tests leads to a reverse conclusion.
To explain above Reversal paradox, consider the following table:

		Manufacturer A	Manufacturer B
Shipment	I	600 good 500 defective	400 good 300 defective
	II	300 good 600 defective	500 good 900 defective

- If the quality control engineer inspects shipment I, he will find:
P(selecting a defective chip from A) = 5/11
> P(selecting a defective chip from B) = 3/7.

- If he then considers shipment II, again
  P(selecting a defective chip from A) = 6/9
  > P(selecting a defective chip from B) = 9/14.

- On Merging the 2 shipments,
  a subsequent test is carried out which lead to a reverse conclusion,

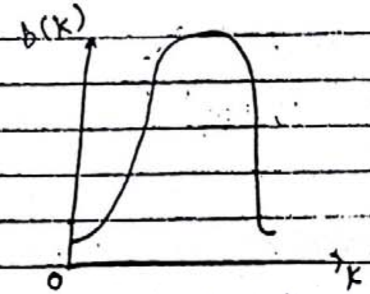
P(selecting a defective chip from A) = 11/20
< P(selecting a defective chip from B) = 12/21.

→ The problem here is that we are tempted to add the fractions $5/11 + 6/9$ and compare the sum with $3/7 + 9/14$; unfortunately, what is called for is adding numerators & adding denominators, which is not the way we add fractions.

Types of Binomial pmf:

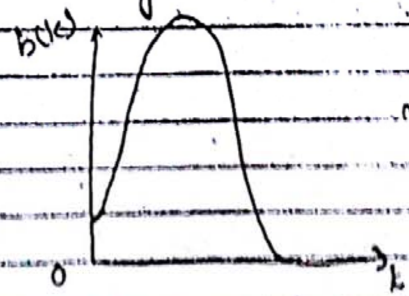
1. Symmetric Binomial pmf
   when $n=5$, $p=0.5$

$$B(k;n,p) = \sum_{i=0}^{k} (b(i;5,0.5))$$



$$P = \binom{5}{k} (0.5)^{k} (0.5)^{\bar{k}}$$

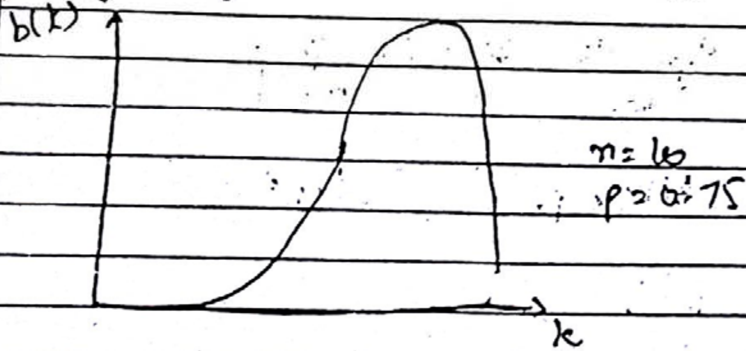
2. Positively skewed binomial pmf



$n=10$, $p=0.25$

$$P = \binom{10}{k} (0.25)^{k} (0.75)^{10-k}$$

3. Negatively skewed binomial pmf



$$p = \binom{10}{k} (0.75)^k (0.25)^{10-k}$$

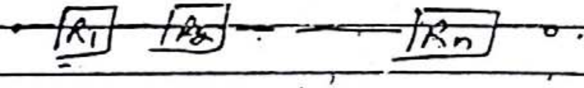
$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\boxed{P(A \cap B) = P(A/B)\, P(B)}$$

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

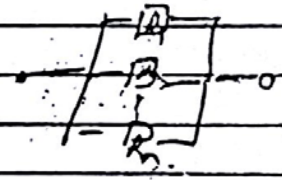
Reliability: Prob. of working.

$R_1 + F_1 = 1$. $0.34 + 0.66 = 1$.



$$R_s = R_1 \cdot R_2 \ldots R_n$$
$$= \prod_{i=1}^{n} R_i$$

Product law of reliability R^n



$$F_p = F_1 \cdot F_2 \ldots F_n$$
$$= (1-R_1)(1-R_2) \ldots (1-R_n).$$
$$= \prod_{i=1}^{n} (1-R_i).$$

$$R_p + F_p = 1$$
$$R_p = 1 - F_p.$$

$$R_p = 1 - \prod_{i=1}^{n} (1-R_i)$$

Product law of unreliability

$$1 - (1-R)^n$$

UNIT-2,

UNIT-2 IInd Part

CONTINUOUS RANDOM VARIABLES

pdf : probability density function [f]

$$f_x(n) = \frac{d}{dx} F_x(n)$$

PDF : probability distribution function

$$P(X \le n) = F_x(n) = \int_{-\infty}^{x} f_x(n)\, dx \longrightarrow \text{atleast}$$

$$P(X > t) = 1 - F_x(t) = \int_{t}^{\infty} f_x(n)\, dx \longrightarrow \text{atmost}$$

$$P(a \le x \le b) = F_x(b) - F_x(a)$$

$$= \int_{a}^{b} f_x(x)\, dx \longrightarrow \text{range}$$

Exponential Distribution

Following are some of the random variables which can be modeled as exponential :-

1.) Lifetime of a component
2.) Time b/w two successive job arrivals to a computing center
3.) Time required to repair a component that has malfunctioned

→ $F_X(t) = 1 - e^{-\lambda t}$; $t > 0$

→ $f_X(t) = \frac{d}{dt}[F_X(t)] = \frac{d}{dt}(1 - e^{-\lambda t})$

$$\boxed{f_X(t) = \lambda e^{-\lambda t}}$$

→ $P(X > t) = \int_t^{\infty} \lambda e^{-\lambda t}\, dt$

or we can use this formula

$1 - F_X(t)$

$= \cancel{1} - \cancel{1} + e^{-\lambda t} = e^{-\lambda t}$

→ $P(a < x \leq b) = \int_a^b \lambda e^{-\lambda t}\, dt$

$= F_X(b) - F_X(a)$

$= \cancel{1} - e^{-\lambda b} - \cancel{1} + e^{-\lambda a}$

$= e^{-\lambda a} - e^{-\lambda b}$

Exponential Distribution has markov property.

$$X = y + t \quad ; x > t$$

$X \rightarrow$ Lifetime of a component
$y \rightarrow$ remaining lifetime
$t \rightarrow$ Time (hrs) that component has already been operated.

$$G = P(Y \leq y \mid x > t)$$

$$= P(X - t \leq y \mid x > t)$$

$$= P(x \leq y + t \mid x > t)$$

$$= \frac{P(t \leq x \leq y+t)}{P(x > t)}$$ [By the defn of conditional Probability]

$$= \frac{\int_t^{y+t} f_X(x)\, dx}{\int_t^{\infty} f_X(x)\, dx} = \frac{e^{-\lambda t} - e^{-\lambda(y+t)}}{e^{-\lambda t}}$$

$$= \frac{\cancel{e^{-\lambda t}}(1 - e^{-\lambda y})}{\cancel{e^{-\lambda t}}}$$

$$= 1 - e^{-\lambda y}$$

eg 3.3 Pg 118 $P(X \geq 10)$

$$F_X(X>10) = \int_{10}^{\infty} (0.1)\, e^{-0.1t}\, dt$$

$$= 1 - F_X(10)$$

$$= \cancel{1} - \cancel{1} + e^{-0.1 \times 10}$$

$$= e^{-1} = 0.368$$

Erlang and Gamma Distribution

Erlang Distribution :-

It has r sequential phases

Each having exponential distribution

Independent and Identical Exponential Distribution

$$\boxed{f(t) = \frac{\lambda^r\, t^{r-1}\, e^{-\lambda t}}{(r-1)!}}$$

* Exponential is the special case of Erlang ~~& Gamma~~ Distribution

$$F(t) = 1 - \sum_{k=0}^{r-1} \frac{(\lambda t)}{k!} e^{-\lambda t}$$

→ ~~Hazard function~~

~~$F(t) = 1-$~~

Gamma Distribution :-

$$\frac{\lambda^{\alpha} t^{\alpha-1} e^{-\lambda t}}{\underbrace{(\alpha-1)!}_{\text{known as Gamma fn}}} = \frac{\lambda^{\alpha} t^{\alpha-1} e^{-\lambda t}}{\underbrace{\Gamma\alpha}_{\text{Gamma}}}$$

{ simply replace i by 'α' in Erlang dist^n for Gamma }

$$\boxed{\Gamma\alpha = (\alpha-1)!}$$

$$\boxed{\int_0^{\infty} x^{\alpha-1} e^{-\lambda x} = \frac{\Gamma\alpha}{\lambda^{\alpha}}}$$ imp.

$Gam(\lambda, \alpha)$

where $\lambda \rightarrow$ scale parameter

$\alpha \rightarrow$ shape parameter

so; $Gam(\lambda, 1) = EXP(\lambda)$

single phase $\rightarrow$ Exponential &

multiple " $\rightarrow$ Erlang.

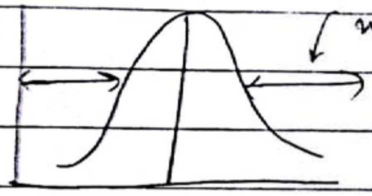
Hyper Exponential Distribution

It has 'k' alternate phases, having exponential distribution independent and non-identical distributions

$$\boxed{f(t) = \sum_{i=0}^{\infty} \alpha_i \lambda_i e^{-\lambda_i t}}$$

$$\boxed{F(t) = \sum_i \alpha_i (1 - e^{-\lambda_i t})}$$

Normal Distribution



not accurate

$\mu \rightarrow$ mean
$\sigma \rightarrow$ standard deviation
$\sigma^2 \rightarrow$ variance

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left[\frac{x-\mu}{\sigma}\right]^2}}$$

$N(\mu, \sigma^2)$

If $N(0,1) \leftarrow$ Standard Normal Distribution & it is denoted by 'Z' and simple N D is
" " 'x'.

$\cancel{Z(0,1)}$ $\quad Z \in N(0,1)$

{putting $\mu = 0$ & $\sigma = 1$}

$$\boxed{f_Z(t) = \frac{1}{\sqrt{2\pi}} e^{-t/2}}$$

$$F_X(t) = \int_{-\infty}^{t} f_Z(t)\,dt$$

$$= \int_{\infty}^{t} f_Z(-t)\,dt$$

$$= \int_{\infty}^{t} f_Z(t)\,dt$$

$$\boxed{F_Z(t) = 1 - F_Z(t)}$$

* $Z = \dfrac{X-\mu}{\sigma}$ $\left\{ \text{If } \mu = 0 \text{ \& } \sigma = 1 \text{ Then } Z = X \right\}$

Eg 3.5
Pg 133

(i) $P(X > 240) = 1 - F_X(240)$

$$= 1 - F_Z\left(\frac{240-200}{16}\right)$$

$$= 1 - F_Z(2.5) \quad \text{[from Pg 591 Table]}$$

$$= 1 - 0.9938$$

$$= 0.00627$$

(ii) $P(X > 240 \mid X > 210) = \dfrac{P(X > 240 \text{ \& } X > 210)}{P(X > 210)}$

$$= \frac{P(X > 240)}{P(X > 210)} = \frac{1 - F_X(240)}{1 - F_X(210)}$$

Eg 3.6
Pg 35

$\mu = 10000 \qquad \sigma = 1000$

$$P(X > 9500 \mid X > 9000) = \frac{P(X > 9500 \;\&\; X > 9000)}{P(X > 9000)}$$

$$= \frac{1 - F_X(9500)}{1 - F_X(9000)} = \frac{1 - F_Z\left(\frac{9500 - 10000}{1000}\right)}{1 - F_Z\left(\frac{9000 - 10000}{1000}\right)}$$

$$= \frac{1 - 1 + F_Z(0.5)}{1 - 1 + F_Z(1)} = \frac{F_Z(0.5)}{F_Z(1)} \text{ Ans.}$$

$$= 0.821 \text{ Ans}$$

Functions of a Random Variable.

$$Y = \phi(X)$$

Eg 3.8 $Y = X^2$

$$F_Y(y) = P(Y \le y)$$
$$= P(X^2 \le y)$$
$$= P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[F_X(\sqrt{y}) - F_X(-\sqrt{y})\right]$$

$$= \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

Ex. class If x has a standard R.D.

$$f_x(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/2}$$

Theorem 3.1 If y is a function of x, find the value of

$$y = \phi(x)$$

$$F_Y(y) = P(Y \le y)$$
$$= P(\phi(x) \le y)$$
$$= P(X \le \phi^{-1}(y))$$
$$= F_X(\phi^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\phi^{-1}(y))$$

$$= f_X(\phi^{-1}(y)) \cdot \frac{d}{dy} \phi^{-1}(y)$$

$$= \phi^{-1\prime}(y)\, f_X(\phi^{-1}(y))$$

Jointly distributed Random variables

Sometimes it is important to consider two or more random variables defined on same probability space.
Let X & Y be two random variables defined on same probability space

$$\boxed{P(X \le x, Y \le y) = F_{X,Y}(x,y)}$$

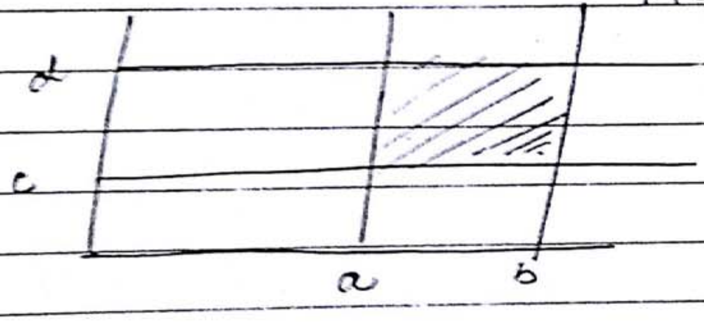
Properties :-

1.) $0 \le F(x,y) \le 1$

2.) $\lim\limits_{x \to -\infty} F(x,y) = 0$; $\lim\limits_{x \to +\infty} F(x,y) = 1$

3.) It is monotone non decreasing function

4.) $P(a \le X \le b ; c < Y < d) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$



$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\,dx\,dy$$

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\, \partial y}$$

Marginal

$$F_X(x) = \int_{-\infty}^{x} \int_{-\infty}^{x} f(x,y)\,dy\,dx$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Independence

$$P(A \cap B) = P(A) \cdot P(B)$$
$$F(x,y) = F(x) \cdot F(y)$$
$$f(x,y) = f(x) \cdot f(y)$$

Chapter 4. EXPECTATION

Expectation, $E[X]$; of a random variable x; is defined by

$$E[X] = \begin{cases} \sum_i x_i \, P(x_i) \;; & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x)\,dx \;; & \text{if } x \text{ is continuous} \end{cases}$$

ie $E[X] = 0 \cdot P_X(0) + 1 \cdot P_X(1) + 2 \cdot P_X(2) + \cdots$

$= \sum_i x_i P_X(x_i) \longrightarrow$ Discrete

& $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \longrightarrow$ continuous

Expectation of Exponential

$$E[X] = \int_0^\infty x \, \lambda e^{-\lambda x} \, dx$$

$$= \lambda \int_0^\infty x e^{-\lambda x} \, dx \quad \text{--- (1)}$$

$\left\{ \text{comparing with } \int_0^\infty x^{\alpha-1} e^{-\lambda x} = \frac{\Gamma\alpha}{\lambda^2} \right.$

$$= \lambda \int_0^\infty x e^{-\lambda x} \, dx$$

$$= \frac{1 \cdot \Gamma 2}{\lambda^2} = \frac{1}{\lambda}$$

* Expectation of Y if $Y = \phi(x)$

$$E[Y] = E[\phi(x)] = \sum_i \phi(x_i) P_X(x_i) \quad \{\text{Discrete}\}$$

$$= \int_{-\infty}^{\infty} \phi(x) f_X(x) \, dx \quad \{\text{continuous}\}$$

* If $Y = X^k \Rightarrow k^{th}$ moment of X

If $x^k = 1 \Rightarrow$ then $y = 1$ & then it become 1st movement of x & the formula for the same is (1)

Variance:- Variance of a Random variable can be defined as

$$Var[X] = \sigma^2 = \begin{cases} \int_{-\infty}^{\infty} (x - E[X])^2 f(x)\,dx; & \text{if } x \text{ is continuous} \\ \Sigma (x - E[X])^2 p(x_i); & \text{if } x \text{ is discrete} \end{cases}$$

If square root of σ (variance) is known as standard deviation

If $Y = (X - E[X])^2$

$$\sigma^2 = E[Y] = E[X - E[X]]^2$$

$$= \sum_i (x_i - E[X])^2 P_X(x_i)$$

for discrete

$$= \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\,dx$$

variance of E[X]

$$\Rightarrow \sigma^2 = \int_0^{\infty} \underbrace{\left(x - \frac{1}{\lambda}\right)^2}_{I} \cdot \underbrace{\lambda e^{-\lambda x}}_{II}\,dx$$

$$\sigma^2 = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\, dx + \int_0^{\infty} \frac{1}{\lambda^2} \lambda e^{-\lambda x}\, dx - \frac{2xe^{-\lambda x}}{\lambda}$$

$$\boxed{\sigma^2 = 1/\lambda^2}$$

Theorem 4.1 The linearity property of Expectation.

Let X & Y be the two random variables, Then the expectation of their sum is the sum of their expectations; that is

if $Z = X+Y$, then $E[Z] = E[X+Y] = E[X] + E[Y]$

$$E[X+Y] = E[X] + E[Y]$$

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f(x,y)\, dy}_{\text{Marginal density}}\, dx + \int_{-\infty}^{\infty} y \underbrace{\int_{-\infty}^{\infty} f(x,y)\, dx}\, dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx + \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$= E[X] + E[Y]$$

$\Rightarrow \quad E[X+Y] = E[X] + E[Y]$

similarly; from variance

$\boxed{\sigma^2 = E[X - E[X]]^2}$ formula 1 of variance

$= E[x^2 + (E[X])^2 - 2 \cdot X \cdot E[X]]$

$= E[x^2] + E[(E(X))^2] - E[2 \cdot X \cdot E[X]]$

$= E[x^2] + (E[X])^2 - 2[E E[X]]^2$

$\boxed{\sigma^2 = E[x^2] - (E[X])^2}$ formula 2 of variance

NOTE: if $E[X \cdot E[X]]$

only expectation of X will be done, E[X] will not be considered.

& if $E[E[X]]$

it will be ~~same~~

Theorem 4.2 $\quad E[X \cdot Y] = E[X] \cdot E[Y]$

If X & Y are two independent random variables.

proof: $E[XY] = \sum_i \sum_j x_i y_j \, P(x_i y_j)$

$= \sum_i \sum_j x_i y_j \, P_X(x_i) P_Y(y_j)$

$$= \sum_i x_i P_X(x_i) \sum_j y_j P_Y(y_j)$$

$$= E[X]E[Y]$$

$$\Rightarrow \boxed{E[XY] = E[X]E[Y]}$$

Theorem 4.3 $\quad var[X+Y] = var[X] + var[Y]$

$$var[X+Y] = E[((X+Y) - E(X+Y))^2]$$

$$= E[(X+Y - E[X] - E[Y])^2]$$

$$= E[[(X-E(X)) + (Y-E(Y))]^2 + 2(X-E[X])(Y-E[Y])]$$

$$= E[(X-E[X])^2] + E[(Y-E[\cancel{X}])^2] + 2E[XY - XE[Y] - YE[X] + E[X]E[Y]]$$

$$= VAR[X] + VAR[Y] \cancel{+} 0$$

$$= VAR[X] + VAR[Y]$$

$$\Rightarrow \boxed{VAR[X+Y] = VAR[X] + VAR[Y]}$$

Uncorrelated Random Variable

Two Random Variables X & Y are said to be uncorrelated provided covariance of X, Y is equal to 0 $\{$ i.e $[X, Y] = 0 \}$

Coefficient of variance (C_X)

$$\boxed{C_X = \frac{\sigma X}{E[X]}}$$

Corelation Coefficient (ρ)

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{var[X]\, var[Y]}}$$

$$= \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

Expectation & variance of Discrete Distribution

1) Discrete uniform Distribution

$$P_X(i) = \frac{1}{n}$$

$$E[X] = \sum_{i=1}^{n} i . P_X(i)$$

$$= \sum_{i=1}^{n} i . \frac{1}{n}$$

$$= \frac{1}{n} \sum_{i=1}^{n} i$$

$$= \frac{n(n+1)}{2} \cdot \frac{1}{n} = \frac{n+1}{2}$$

$$\boxed{E[X] = \frac{n+1}{2}}$$

$$var[X] = E[X^2] - [E(X)]^2$$

$$E[X^2] = \sum_{i=1}^{n} i^2 P_X(i)$$

$$= \sum_{i=1}^{n} \frac{i^2}{n} = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{\cancel{n}(n+1)(2n+1)}{\cancel{n}\ 6}$$

$$\therefore var[X] = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}$$

$$C_X = \frac{\sigma_X}{E[X]} = \frac{\sqrt{\frac{n^2-1}{12}}}{\frac{(n+1)}{12}}$$

2) Expectation & variance of Bernoulli pmf

$$E[X] = 0 \cdot P_X(0) + 1 \cdot P_X(1) = 0 \cdot q + 1 \cdot p = p$$

$$E[X^2] = 0^2 \cdot P_X(0) + 1^2 \cdot P_X(1) = p$$

$$var[X] = E[X^2] + [E[X]]^2$$

$$= p - p^2 \quad p(1-p) = pq$$

$$= pq$$

$$C_x = \frac{\sqrt{pq}}{p} \sqrt{q/p}$$

3) Binomial Pmf :- "n independent Bernoulli trials from binomial pmf."

$$E[X] = np$$

$$var[X] = npq$$

$$C_x = \frac{\sqrt{npq}}{np} = \sqrt{q/np}$$

4) Geometric pmf :- $E[X] = \sum_{i=1}^{\infty} ipq^{i-1}$

$$= p \sum_{i=1}^{\infty} iq^{i-1}$$

$$= p \cdot \sum_{i=0}^{\infty} \frac{d}{dq}(q^i)$$

$$= p \frac{d}{dq} \sum_{i=0}^{\infty} q^i$$

$$= p \frac{d}{dq} \left[\frac{1}{1-q}\right]$$

$$= \frac{p}{(1-q)^2} = \frac{p}{p^2} = 1/p$$

$$\mathrm{var}(x) = q/p^2$$

$$C_x = \sqrt{\frac{q/p^2}{1/p}} = \sqrt{q} = \sqrt{1-p}.$$

5) Poisson Pmf :-

$$E[X] = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\alpha}(\alpha)^k}{k!}$$

$$= e^{-\alpha} \sum_{k=0}^{\infty} \frac{k(\alpha)^k}{k!}$$

$$= e^{-\alpha} \sum_{k=0}^{\infty} \frac{k(\alpha)^k}{k(k-1)}$$

$$= e^{-\alpha}\alpha \sum_{k=0}^{\infty} \frac{\alpha^{k-1}}{(k-1)!}$$

$$= \alpha e^{-\alpha} \cdot e^{\alpha} = \alpha$$

$$\mathrm{var}[X] = E[X^2] - [E[X]]^2 = \alpha$$

$$C_\alpha = \frac{\sqrt{\alpha}}{\alpha} = \frac{1}{\sqrt{\alpha}}$$

Expectation and variance of continuous distribution

1) Continuous uniform distribution

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x)$$

$$= \int_a^b \frac{x}{b-a} \, dx = \frac{1}{b-a} \int_a^b x \, dx$$

$$= \frac{1}{b-a} \left| \frac{x^2}{2} \right|_a^b$$

$$= \frac{1}{b-a} \left| \frac{x^2}{2} \right|_a^b$$

$$= \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{b+a}{2}$$

$$\text{var}[X] = E[X^2] - [E[X]]^2$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) \, dx = \int_a^b \frac{x^2}{b-a} \, dx$$

$$= \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b$$

$$\frac{1}{ba}\left[\frac{b-a^3}{3}\right]$$

$$\Rightarrow E[x^2] = \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right] = \frac{b^3-a^3}{3(b-a)}$$

$$\mathrm{var}[x] = E[x^2] - [E[x]]^2$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{(b+a)^2}{4}$$

$$= \frac{4(b^3-a^3) - 3(b-a)(b+a)^2}{3\cdot 4(b-a)} = \frac{4(b^3-a^3) - 3[(b-a)(b^2+a^2+2ab)]}{12(b-a)}$$

✓ $$\Rightarrow \mathrm{var}[x] = \frac{(b-a)^2}{12}$$

✓ $$C_x = \frac{b-a}{b+a}$$

2) <u>Exponential distribution</u>

$$E[x] = \int_{-\infty}^{\infty} x f_x(x)$$

$$= \int_0^{\infty} x\lambda e^{-\lambda x}\,dx$$

$$= \lambda\int_0^{\infty} x \cdot e^{-\lambda x}\,dx = \frac{\lambda \cdot \sqrt{2}}{\lambda^2} = 1\cdot\frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$\mathrm{var}[x] = E[x^2] - [E[x]]^2;\ E[x^2] = \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x}\,dx = \frac{2}{\lambda^2}$$

3) Expectation of Gamma Distribution

$$f_X(t) = \frac{\lambda^{\alpha} t^{\alpha-1} e^{-\lambda t}}{(\alpha-1)!}$$

$$E[X] = \int_0^{\infty} t \cdot f_X(t)\,dt$$

$$= \int_0^{\infty} \frac{\lambda^{\alpha} \cdot t^{\alpha} e^{-\lambda t}}{(\alpha-1)!}\,dt$$

$$= \frac{\lambda^{\alpha}}{(\alpha-1)!} \int_0^{\infty} t^{\alpha} \cdot e^{-\lambda t}\,dt$$

$$= \frac{\alpha!}{(\alpha-1)!} = \frac{1}{\lambda} = \frac{\alpha}{\lambda}$$

$$E[X^2] = \int_0^{\infty} t^2 f_X(t)\,dt$$

$$= \int_0^{\infty} \frac{\lambda^{\alpha} t^{\alpha+1} e^{-\lambda t}\,dt}{(\alpha-1)!}$$

$$= \frac{\lambda\alpha}{(\alpha-1)!} \int_0^{\infty} t^{\alpha+1} e^{-\lambda t}\,dt$$

$$\boxed{E[X] = \frac{\alpha(\alpha+1)}{\lambda^2}}$$

$$var[X] = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{\alpha^2 + \alpha - \alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

$$\Rightarrow \boxed{var[X] = \frac{\alpha}{\lambda^2}}$$

$$\boxed{C_X = \frac{1}{\sqrt{\alpha}}}$$

MEAN TIME TO FAILURE :—

$$F(t) = 1 - R(t)$$

Reliability

$$R(t) = 1 - F(t)$$

on Taking derivative w.r.t. 't'

$$R'(t) = -f(t)$$

$$f(t) = -R'(t)$$

$$\boxed{R(t) = P(X > t)}$$

$$E[x] = \int_0^\infty t\, f(t)\, dt$$

$$= -\int_0^\infty t \cdot R'(t)\, dt$$

$$\Rightarrow E[x] = -tR(t)\Big|_0^\infty + \int_0^\infty R(t)\, dt$$

Before 't' reaches to ∞, $R(t)$ becomes zero.

$$\Rightarrow E[x] = \int_0^\infty R(t)\, dt$$

$$E[x^k] = \int_0^\infty f(t)\, dt = \int_0^\infty t^k R(t)\, dt$$

$$= -t^k R(t)\Big|_0^\infty + \int_0^\infty k t^{k-1} R(t)\, dt$$

$$= \int_0^\infty k t^{k-1} R(t)\, dt \quad \text{——— (A)}$$

$$\text{Var}[x] = E[x^2] - [E[x]]^2$$

$$E[x^2] = \int_0^\infty 2t\, R(t)\, dt$$

$$\mathrm{var}[x] = E[x^2] - [E[x]]^2$$

$$= \int_0^\infty 2t\, R(t)\, dt - \left[\int_0^\infty R(t)\, dt\right]^2$$

$R(t) = 1 - F(t)$ &

$\because$ $F(t) = 1 + e^{-\lambda t}$

$= 1 - 1 + e^{-\lambda t} \Rightarrow \boxed{R(t) = e^{-\lambda t}}$

$$E[x] = \int_0^\infty e^{-\lambda t}\, dt$$

$\Rightarrow$ $= \frac{\pi}{\lambda} = \frac{1}{\lambda} \Rightarrow E[x] = 1/\lambda$

$$(E[x])^2 = \int_0^t 2t\, e^{-\lambda t}\, dt$$

from value of $R(t) = e^{-\lambda t}$

$$\mathrm{var}[x] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Mean time to failure in series system:-

$$R(t) = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{\left[\sum_{i=1}^{n} \lambda t\right] t}$$

So MTTF

$$= \frac{1}{\sum_{i=1}^{n} \lambda_i}$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdot e^{-\lambda_3 t} \cdots e^{-\lambda_n t}$$

$$= \prod_{i=1}^{n} e^{-\prod_{i=1}^{n} \lambda_i t}.$$

Mean time to failure for a Parallel system

$$R(t) = e^{-\lambda t}$$

$$R(t) = 1 - (1 - e^{-\lambda t})^n$$

$$E[x] = \int_0^\infty \left[1 - (1 - e^{-\lambda t})^n\right] dt$$

Assume, $u = 1 - e^{-\lambda t}$

$$\frac{du}{dt} = 0 - (-\lambda) e^{-\lambda t}$$

$$\frac{du}{dt} = \lambda e^{-\lambda t}$$

$$\frac{du}{dt} = \lambda(1-u)$$

$$dt = \frac{du}{\lambda(1-u)}$$

Also $t \to 0$

$u \to 0$

$t \to \infty$

$u \to 1$

$$E[X] = \frac{1}{\lambda}\int_0^{\infty} \frac{1-u^n}{1-u}\, du$$

$$= \frac{1}{\lambda}\int_0^1 \left(\sum_{i=1}^{n} u^{i-1}\right) du$$

$$= \frac{1}{\lambda}\int_0^1 \left(\sum_{i=1}^{n} u^{i-1}\right) du$$

$$= \frac{1}{\lambda}\sum_{i=1}^{n-1} \frac{1}{i} = \frac{\ln(n)}{\lambda}$$